PROBLEMS FOR APRIL

Please send your solution to
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no later than May 31, 2003. It is important that your complete mailing address and your email address appear on the front page.

Notes. Let x be a real number. The *inverse tangent function*, $\tan^{-1} x$ (sometimes referred to as $\arctan x$) is that number θ for which $-\pi/2 < \theta < \pi/2$ and $\tan \theta = x$.

220. Prove or disprove: A quadrilateral with one pair of opposite sides and one pair of opposite angles equal is a parallelogram.

221. A *cycloid* is the locus of a point P fixed on a circle that rolls without slipping upon a line u. It consists of a sequence of arches, each arch extending from that position on the locus at which the point P rests on the line u, through a curve that rises to a position whose distance from u is equal to the diameter of the generating circle and then falls to a subsequent position at which P rests on the line u. Let v be the straight line parallel to u that is tangent to the cycloid at the point furthest from the line u.

(a) Consider a position of the generating circle, and let P be on this circle and on the cycloid. Let PQ be the chord on this circle that is parallel to u (and to v). Show that the locus of Q is a similar cycloid formed by a circle of the same radius rolling (upside down) along the line v.

(b) The region between the two cycloids consists of a number of "beads". Argue that the area of one of these beads is equal to the area of the generating circle.

(c) Use the considerations of (a) and (b) to find the area between u and one arch of the cycloid using a method that does not make use of calculus.

222. Evaluate

$$\sum_{n=1}^{\infty} \tan^{-1} \left(\frac{2}{n^2} \right) .$$

223. Let a, b, c be positive real numbers for which $a + b + c = abc$. Prove that

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2} .$$

224. For $x > 0$, $y > 0$, let $g(x, y)$ denote the minimum of the three quantities, x, $y + 1/x$ and $1/y$. Determine the maximum value of $g(x, y)$ and where this maximum is assumed.

225. A set of n lighbulbs, each with an *on-off* switch, numbered $1, 2, \cdots, n$ are arranged in a line. All are initially off. Switch 1 can be operated at any time to turn its bulb on of off. Switch 2 can turn bulb 2 on or off if and only if bulb 1 is off; otherwise, it does not function. For $k \geq 3$, switch k can turn bulb k on or off if and only if bulb $k - 1$ is off and bulbs $1, 2, \cdots, k - 2$ are all on; otherwise it does not function.

(a) Prove that there is an algorithm that will turn all of the bulbs on.

(b) If x_n is the length of the shortest algorithm that will turn on all n bulbs when they are initially off, determine the largest prime divisor of $3x_n + 1$ when n is odd.

226. Suppose that the polynomial $f(x)$ of degree $n \geq 1$ has all real roots and that $\lambda > 0$. Prove that the set $\{x \in \mathbf{R} : |f(x)| \leq \lambda |f'(x)|\}$ is a finite union of closed intervals whose total length is equal to $2n\lambda$.