PROBLEMS FOR JANUARY

Please send your solution to
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no later than February 21, 2003. It is important that your complete mailing address and your email address appear on the front page.

Notes. A function is *convex* if and only if for each u and v, and for each $t \in [0,1]$, $f(tu + (1-t)v) \leq tf(u) + (1-t)f(v)$.

199. Let A and B be two points on a parabola with vertex V such that VA is perpendicular to VB and θ is the angle between the chord VA and the axis of the parabola. Prove that

$$\frac{|VA|}{|VB|} = \cot^3 \theta \ .$$

200. Let n be a positive integer exceeding 1. Determine the number of permutations $(a_1, a_2, \cdots, a_n)$ of $(1, 2, \cdots, n)$ for which there exists exactly one index i with $1 \leq i \leq n$ and $a_i > a_{i+1}$.

201. Let $(a_1, a_2, \cdots, a_n)$ be an arithmetic progression and $(b_1, b_2, \cdots, b_n)$ be a geometric progression, each of n positive real numbers, for which $a_1 = b_1$ and $a_n = b_n$. Prove that

$$a_1 + a_2 + \cdots + a_n \geq b_1 + b_2 + \cdots + b_n \ .$$

202. For each positive integer k, let $a_k = 1 + (1/2) + (1/3) + \cdots + (1/k)$. Prove that, for each positive integer n,

$$3a_1 + 5a_2 + 7a_3 + \cdots + (2n+1)a_n = (n+1)^2 a_n - \frac{1}{2}n(n+1) \ .$$

203. Every midpoint of an edge of a tetrahedron is contained in a plane that is perpendicular to the opposite edge. Prove that these six planes intersect in a point that is symmetric to the centre of the circumsphere of the tetrahedron with respect to its centroid.

204. Each of $n \geq 2$ people in a certain village has at least one of eight different names. No two people have exactly the same set of names. For an arbitrary set of k names, where $1 \leq k \leq 7$, the number of people containing at least one of the k names among his/her set of names is even. Determine the value of n.

205. Let $f(x)$ be a convex realvalued function defined on the reals, $n \geq 2$ and $x_1 < x_2 < \cdots < x_n$. Prove that

$$x_1 f(x_2) + x_2 f(x_3) + \cdots + x_n f(x_1) \geq x_2 f(x_1) + x_3 f(x_2) + \cdots + x_1 f(x_n) \ .$$