PROBLEMS FOR MAY

Please send your solution to
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no later than June 30, 2003. It is important that your complete mailing address and your email address appear on the front page.

Notes: The notation $[a, b]$ refers to the closed interval $\{x : a \leq x \leq b\}$. A function f defined on a closed interval is *strictly increasing* iff $f(u) < f(v)$ whenever $u < v$. Such a function has an inverse function g defined on the image of f that satisfies $y = g(x)$ if and only if $x = f(y)$.

227. Let n be an integer exceeding 2 and let $a_0, a_1, a_2, \cdots, a_n, a_{n+1}$ be positive real numbers for which $a_0 = a_n$, $a_1 = a_{n+1}$ and
$$a_{i-1} + a_{i+1} = k_i a_i$$
for some positive integers k_i, where $1 \leq i \leq n$.

Prove that
$$2n \leq k_1 + k_2 + \cdots + k_n \leq 3n \ .$$

228. Prove that, if $1 < a < b < c$, then
$$\log_a(\log_a b) + \log_b(\log_b c) + \log_c(\log_c a) > 0 \ .$$

229. Suppose that n is a positive integer and that $0 < i < j < n$. Prove that the greatest common divisor of $\binom{n}{i}$ and $\binom{n}{j}$ exceeds 1.

230. Let f be a strictly increasing function on the closed interval $[0, 1]$ for which $f(0) = 0$ and $f(1) = 1$. Let g be its inverse. Prove that
$$\sum_{k=1}^{9} \left(f\left(\frac{k}{10}\right) + g\left(\frac{k}{10}\right) \right) \leq 9.9 \ .$$

231. For $n \geq 10$, let $g(n)$ be defined as follows: n is mapped by g to the sum of the number formed by taking all but the last three digits of its square and adding it to the number formed by the last three digits of its square. For example, $g(54) = 918$ since $54^2 = 2916$ and $2 + 916 = 918$. Is it possible to start with 527 and, through repeated applications of g, arrive at 605?

232. (a) Prove that, for positive integers n and positive values of x,
$$(1 + x^{n+1})^n \leq (1 + x^n)^{n+1} \leq 2(1 + x^{n+1})^n \ .$$

(b) Let $h(x)$ be the function defined by
$$h(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1; \\ x, & \text{if } x > 1. \end{cases}$$

Determine a value N for which
$$|h(x) - (1 + x^n)^{\frac{1}{n}}| < 10^{-6}$$
whenever $0 \leq x \leq 10$ and $n \geq N$.

233. Let $p(x)$ be a polynomial of degree 4 with rational coefficients for which the equation $p(x) = 0$ has *exactly one* real solution. Prove that this solution is rational.