Solutions

227. Let n be an integer exceeding 2 and let $a_0, a_1, a_2, \cdots, a_n, a_{n+1}$ be positive real numbers for which $a_0 = a_n$, $a_1 = a_{n+1}$ and

$$a_{i-1} + a_{i+1} = k_i a_i$$

for some positive integers k_i, where $1 \le i \le n$.

Prove that

$$2n \le k_1 + k_2 + \cdots + k_n \le 3n \ .$$

Solution. Since $k_i = (a_{i-1}/a_i) + (a_{i+1}/a_i)$ for each i,

$$\sum_{i=1}^{n} k_i = \sum_{i=1}^{n} \left(\frac{a_{i+1}}{a_i} + \frac{a_i}{a_{i+1}} \right) \ge \sum_{i=1}^{n} 2 = 2n \ .$$

As for the other inequality, since the expression has cyclic symmetry, there is no loss in generality in supposing that $a_n \ge a_1$ and $a_n \ge a_{n-1}$ with inequality in at least one case, so that $2a_n > a_{n-1} + a_1$. Therefore, $k_n = 1$ and $a_n = a_{n-1} + a_1$.

We establish the right inequality by induction. For the case $n = 3$, we may suppose that

$$a_2 + a_3 = k_1 a_1 \ ; \quad a_1 + a_3 = k_2 a_2 \ ; \quad a_1 + a_2 = a_3$$

Substituting for a_3 and rearranging the terms yields the brace of equations

$$2a_2 = (k_1 - 1)a_1 \qquad 2a_1 = (k_2 - 1)a_2$$

whence $4 = (k_1 - 1)(k_2 - 1)$. It follows that $k_1 + k_2 + k_3$ is either $5 + 2 + 1 = 8$ or $3 + 3 + 1 = 7$.

Now suppose the result holds when the index is $n - 1 \ge 3$. Then, supposing that $k_n = 1$ and substituting for a_n, we obtain the $n - 1$ equations

$$a_{n-1} + a_2 = (k_1 - 1)a_1$$

$$a_1 + a_3 = k_2 a_2$$

$$\cdots$$

$$a_{n-3} + a_{n-1} = k_{n-2} a_{n-2}$$

$$a_{n-2} + a_1 = (k_{n-1} - 1)a_{n-1} \ .$$

By the induction hypothesis

$$(k_1 - 1) + k_2 + \cdots + (k_{n-1} - 1) \le 3(n-1) = 3n - 3$$

whence

$$k_1 + k_2 + \cdots + k_n \le (3n - 3) + 2 + 1 = 3n \ .$$

241. Determine $\sec 40^\circ + \sec 80^\circ + \sec 160^\circ$.

Solution 1. The values 40°, 80° and 160° all satisfy $\cos 3\theta = 3D - 1/2$, or $8\cos^3\theta - 6\cos\theta + 1 = 3D0$. Thus, $\cos 40^\circ$. $\cos 80^\circ$ and $\cos 160^\circ$ are the roots of the cubic equation $8x^3 - 6x + 1 = 3D0$, so that their reciprocals $\sec 40^\circ$, $\sec 80^\circ$ and $\sec 160^\circ$ are the roots of the cubic equation $x^3 - 6x^2 + 8 = 3D0$. The sum of the roots of this cubic is

$$\sec 40^\circ + \sec 80^\circ + \sec 160^\circ = 3D6 \ .$$

Solution 2. Let $z = 3D\cos 40^\circ + i\sin 40^\circ$. Then $z^9 = 3D1$. In fact, since $z^9 - 1 = 3D(z-1)(z^2+z+1)(z^6+z^3+1)$ and the first two factors fail to vanish, $z^6 + z^3 + 1 = 3D0$. Also $1 + z + z^2 + \cdots + z^8 = 3D(1+z+z^2)(1+z^3+z^6) = 3D0$. Observe that $\cos 40^\circ = 3D\frac{1}{2}(z + \frac{1}{z})$, $\cos 80^\circ = 3D\frac{1}{2}(z^2 + \frac{1}{z^2})$ and $\cos 160^\circ = 3D\frac{1}{2}(z^4 + \frac{1}{z^4})$, so that the given sum is equal to

$$\begin{aligned} 2\left[\frac{z}{1+z^2} + \frac{z^2}{1+z^4} + \frac{z^4}{1+z^8}\right] &= 3D2\left[\frac{z}{1+z^2} + \frac{z^2}{1+z^4} + \frac{z^5}{1+z}\right] \\ &= 3D2\left[\frac{z(1+z+z^4+z^5) + z^2(1+z+z^2+z^3) + z^5(1+z^2+z^4+z^6)}{(1+z)(1+z^2)(1+z^4)}\right] \\ &= 3D2\left[\frac{z^7+z^6+3z^5+z^4+z^3+3z^2+z+1}{(1+z)(1+z^2)(1+z^4)}\right] \\ &= 3D2\left[\frac{(z+1)(z^6+z^3+1)+3z^2(z^3+1)}{(1+z)(1+z^2)(1+z^4)}\right] \\ &= 3D2\left[\frac{0-3z^8}{1+z+z^2+z^3+z^4+z^5+z^6+z^7}\right] = 3D2\left[\frac{-3z^8}{-z^8}\right] = 3D6\ . \end{aligned}$$

Solution 3. [T. Liu]

$$\begin{aligned} \sec 40^\circ + \sec 80^\circ + \sec 160^\circ &= 3D\frac{\cos 40^\circ + \cos 80^\circ}{\cos 40^\circ \cos 80^\circ} + \frac{1}{\cos 160^\circ} \\ &= 3D\frac{2\cos 60^\circ \cos 20^\circ}{\cos 40^\circ \cos 80^\circ} + \frac{1}{\cos 160^\circ} \\ &= 3D\frac{\cos 20^\circ \cos 160^\circ + \cos 40^\circ \cos 80^\circ}{\cos 40^\circ \cos 80^\circ \cos 160^\circ} \\ &= 3D\frac{\cos 180^\circ + \cos 140^\circ + \cos 120^\circ + \cos 40^\circ}{\cos 40^\circ(\cos 240^\circ + \cos 80^\circ)} \\ &= 3D\frac{-1-1/2}{(1/2)(-\cos 40^\circ + \cos 120^\circ + \cos 40^\circ)} = 3D\frac{-3/2}{-1/4} = 3D6\ . \end{aligned}$$

Solution 4. Let $x = 3D\cos 40^\circ$, $y = 3D\cos 80^\circ$ and $z = 3D\cos 160^\circ$. Then

$$x + y + z = 3D2\cos 60^\circ \cos 20^\circ - \cos 20^\circ = 3D0$$

and

$$\begin{aligned} xy + yz + zx &= 3D\frac{1}{2}[\cos 120^\circ + \cos 140^\circ + \cos 240^\circ + \cos 80^\circ + \cos 200^\circ + \cos 120^\circ] \\ &= 3D\frac{1}{2}\left[-\frac{3}{2} + x + y + z\right] = 3D - \frac{3}{4}\ . \end{aligned}$$

Now

$$\begin{aligned} 8\sin 40^\circ \cos 40^\circ \cos 80^\circ \cos 160^\circ &= 3D4\sin 80^\circ \cos 80^\circ \cos 160^\circ \\ &= 3D2\sin 160^\circ \cos 160^\circ = 3D\sin 320^\circ = 3D - \sin 40^\circ \end{aligned}$$

so that $xyz = 3D - 1/8$. Then the sum of the problem is equal to $(xy + yz + zx)/(xyz) = 3D6$.

248. Find all real solutions to the equation

$$\sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x + 8 - 6\sqrt{x-1}} = 1\ .$$

Solution 1. For the equation to be valid over the reals, we require that $x \geq 1$. Suppose that $y^2 = x - 1$ and $y \geq 0$. Then the equation becomes

$$|y - 2| + |y - 3| = 1\ .$$

When $1 \leq x \leq 5$, we have that $0 \leq y \leq 2$ and the equation becomes $(2-y)+(3-y)=1$ or $y=2, x=5$. When $5 \leq x \leq 10$, we have that $2 \leq y \leq 3$ and the equation becomes an identity $(y-2)+(3-y)=1$. Thus, it holds for all x on the closed interval $[5,10]$. Finally, when $10 \leq x$, we have that $3 \leq y$ and the equation becomes $(y-2)+(y-3)=1$ or $y=3, x=10$. Thus, the complete set of solutions of the equation is given by $5 \leq x \leq 10$. All these solutions check out.

Solution 2. [Z. Wu] For a solution to exist, we require that $x \geq 1$ and that both $0 \leq x+3-4\sqrt{x-1} \leq 1$ and $0 \leq x+8-6\sqrt{x-1} \leq 1$. These two conditions lead to $(x+2)^2 \leq 16(x-1)$ and $(x+7)^2 \leq 36(x-1)$, which in turn leads to

$$(x-2)(x-10) = (x^2+4x+4) - (16x-16) \leq 0$$

and

$$(x-5)(x-17) = (x^2+14x+49) - (36x-36) \leq 0 \ .$$

These conditions are both satisfied only if $5 \leq x \leq 10$. (Thus, $5 \leq x \leq 10$ is a *necessary* condition for a solution.)

On the other hand, $5 \leq x \leq 10$ implies that $2 \leq \sqrt{x-1} \leq 3$, so that (as in Solution 1) we find that the equation is equivalent to $(\sqrt{x-1}-2)+(3-\sqrt{x-1})=1$, which is an identity. Thus, the equation holds exactly when $5 \leq x \leq 10$.

Comment. Your first observation should be that, in order for the equation to make sense, we require that $x \geq 1$. It is important not just to write down a lot of algebraic equations, but to indicate the logical relationships between them; which equations imply which other equations? which pairs of equations are equivalent? This is especially desirable when surd equations are involved, where the operations that lead from one equation to another are not logically reversible and extraneous solutions might be introduced.

249. The non-isosceles right triangle ABC has $\angle CAB = 90°$. Its inscribed circle with centre T touches the sides AB and AC at U and V respectively. The tangent through A of the circumscribed circle of triangle ABC meets UV in S. Prove that:

(a) $ST \| BC$;

(b) $|d_1 - d_2| = r$, where r is the radius of the inscribed circle, and d_1 and d_2 are the respective distances from S to AC and AB.

Solution. Wolog, we may assume that $AB < AC$ so that S and C are on opposite sides of AB. Ad (a), $\angle SVT = \angle SVA = 45°$, $AV = VT$ and SV is common, so that triangles AVS and TVS are congruent. Hence $\angle SAV = \angle STV \Longrightarrow \angle STU = \angle SAU = \angle ACB$ (by the tangent-chord property). Since $TU \| AC$, it follows that $CB \| ST$.

Ad (b), let P and Q be the respective feet of the perpendiculars from S to AB and AC. Note that $SQAP$ is a rectangle so that $\angle PUS = \angle PSU = 45°$ and so $PU = PS$. Then $|QS| - |PS| = |AP| - |PU| = r$.

250. In a convex polygon $\mathcal{P}$, some diagonals have been drawn so that no two have an intersection in the interior of $\mathcal{P}$. Show that there exists at least two vertices of $\mathcal{P}$, neither of which is an enpoint of any of these diagonals.

Solution 1. If no diagonal has been drawn, the result is clear. Suppose that at least one diagonal has been drawn. Let d be a diagonal that has, on one of its sides, the fewest vertices of the polygon. There is at least one such vertex. Then on that side, no further diagonal is drawn, since it cannot cross d and cannot have fewer vertices between its endpoints than d. Hence there is at least one vertex on that side from which no diagonal is drawn.

On the other side of d, select a diagonal g which has the smallest number of vertices between its endpoints on the side opposite to the side of d. By an argument similar to the above, there is at least one vertex on the side of g opposite to d from which no diagonal has been drawn.

Solution 2. [S. King] The result is vacuously true for triangles. Suppose that the polygon has at least four sides. Suppose that a (possibly void) collection of diagonals as specified in the problem is given. We continue adding diagonals one at a time such that each new diagonal does not cross any previous one in the interior of the polygon. At each stage, the polygon $\mathcal{P}$ is partitioned into polygons with fewer sides all of whose vertices are vertices of the polygon $\mathcal{P}$. As long as any of the subpolygons has more than three sides, we can add a new diagonal. However, the process will eventually terminate with a triangulation of $\mathcal{P}$, *i.e.*, a partitioning of $\mathcal{P}$ into $n-2$ triangles all of whose vertices are vertices of $\mathcal{P}$. (*Exercise.* Explain why the number of triangles is $n-2$. One way to do this is to note that the sum of all the angles of the triangles is equal to the sum of the angles in the polygon.)

Each triangle must have at most two sides in common with the given polygon. Since there are n sides and $n-2$ triangles, at least two triangles have two sides in common with $\mathcal{P}$. In each case, the vertex common to the two sides has no diagonal emanating from it (neither an original diagonal nor an added diagonal), and the result follows.

Comment. Many solvers failed to appreciate that the collection of diagonals is given, and that the problem is to establish the desired property no matter what the collection is. A lot of arguments had the students constructing diagonals without indicating how the ones constructed might have anything to do with a given set; in effect, they were giving a particular situation in which the result obtained. Several solvers tried induction, using one diagonal to split $\mathcal{P}$ into two, but did not handle well the possibility that the loose vertices in the subpolygons might be at the ends of the subdividing diagonal. One way around this is to make the result stronger, and show that one can find two *non-adjacent* vertices that are not the endpoints of diagonals. This is certainly true for quadrilaterals, and using this an induction hupothesis yielded a straightforward argument for polygons of higher order.

251. Prove that there are infinitely many positive integers n for which the numbers $\{1, 2, 3, \cdots, 3n\}$ can be arranged in a rectangular array with three rows and n columns for which (a) each row has the same sum, a multiple of 6, and (b) each column has the same sum, a multiple of 6.

Solution 1. The sum of all the numbers in the array is $3n(3n+1)/2$, so that each column sum must be $3(3n+1)/2$. Since this is divisible by 6, $3(3n+1)$ must be a multiple of 12, and so $3n+1$ is divisible by 4 and $n \equiv 1 \pmod 4$. Since each row sum, $n(3n+1)/2$ is divisible by 6, n must be divisible by 3. Putting this together, we conclude that $n = 12k+9$ for some value of k.

We now show that, for each n of this form, we can actually construct an array with the desired property. Starting with the magic square, we derive the following array for $n = 9$:

8	1	6	17	10	15	26	19	24
21	23	25	3	5	7	12	14	16
13	18	11	22	27	20	4	9	2

We generalize this for $n = 12k+9$, for k a nonnegative integer. Suppose that an array is possible. Then the sum of all the elements in the array is $(36k+27)(18k+14) = 18(4k+3)(9k+7)$. The sum of the elements in each column is $6(9k+7) = 54k+42$ and the sum in each row is $6(4k+3)(9k+7) = (4k+3)(54k+42)$. If we can achieve this with distinct entries, then we have constructed the array.

We build the array by juxtaposing horizontally $4k+3$ square 3×3 blocks of the form:

$8+9a$	$1+9a$	$6+9a$
$3+9b$	$5+9b$	$7+9b$
$4+9c$	$9+9c$	$2+9c$

where we make $4k+3$ distinct choices of each of a, b, c to ensure that no number is repeated in any row (it is not possible for any repetition to occur down a column). To achieve the column sum, we require that $15 + 9(a+b+c) = 54k+42$, or $a+b+c = 6k+3 = 3(2k+1)$. To achieve the row sum, we require that

$$\begin{aligned} 15(4k+3) + 27\sum a &= 15(4k+3) + 27\sum b = 15(4k+3) + 27\sum c \\ &= (4k+3)(54k+42) \end{aligned}$$

so that

$$\sum a = \sum b = \sum c = (4k+3)(2k+1) = 0 + 1 + \cdots + (4k+2) \ ,$$

where each sum is over $4k+3$ distinct elements. It is convenient to let the sets of a's, b's, and c's each consist of the numbers $0, 1, 2, \cdots, 4k+2$ in some order. In the ith 3×3 block, let $0 \le a, b, c, \le 4k+2$ and

$$a = i - 1$$

$$b \equiv (i-1) + 2(k+1) = 2k + i + 1 \pmod{4k+2}$$

$$c = (6k+3) - (a+b) \equiv 4k + 3 - 2i \pmod{4k+3} \ .$$

for $1 \le i \le 4k+3$. It is straightforward to verify that the a's, the b's and the c's each run through a complete set of residues (mod $4k+3$), and we have arranged that $a+b+c = 6k+3$. If $1 \le i \le 2k+1$, then $2k+2 \le b = 2k+i+1 \le 4k+2$ and $2k+2 \le a+b = 2(k+i) \le 6k+2$, so that $1 \le c \le 4k+1$. If $2k+2 \le i \le 4k+3$, then $0 \le b = (2k+i+1) - (4k+3) = i - 2k - 2 \le 2k+1$ and $2k+1 \le a+b = 2i - 2k - 3 \le 6k+3$, so that $0 \le c \le 4k+2$. With this choice of the variables a, b, c we can construct the array as desired.

For example, when $n = 45, k = 3$, there are 15 blocks and the choice of a, b, c for these blocks can be read along the rows of

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
8	9	10	11	12	13	14	0	1	2	3	4	5	6	7
13	11	9	7	5	3	1	14	12	10	8	6	4	2	0

It is left as an exercise for the reader to construct the 3×45 array.

Solution 2. [Y. Zhao] We can form the 3×9 array:

4	9	2	13	18	11	22	27	20
12	14	16	21	23	25	3	5	7
26	19	24	8	1	6	17	10	15

Suppose, as an induction hypothesis, we can build a $3 \times n$ array for some positive integer n. Duplicate this array five times and put them side by side in a row. Partition the $3 \times 5n$ array into fifteen $1 \times n$ subarrays, and to the elements of each of the fifteen subarrays add a constant number as indicated by the positions in the following 3×5 table:

+0	+3b	+6n	+9n	+12n
+6n	+9n	+12n	+0	+3
+12n	+6n	+0	+9n	+3n

The row sum of the numbers added is $30n$ and the column sum is $18n$, so the $3 \times 5n$ array preserves the divisibility by 6 properties of the $3 \times n$ array. Therefore, we can see by induction that an array is constructibel whenever $n = 9 \times 5^k$ for $0 \le k$.

Solution 3. [J. Zhao] For the time being, neglect the conditions involving divisibility by 6, and focus only on the condition that the numbers $1, 2, \cdots, 3n$ be used and that the row sums and the column sums be each the same. Then, when $n = 3$, a magic square will serve.

Suppose that, for some $k \ge 1$, we have found a suitable 3×3^k matrix M. Let A be the $3 \times 3^{k+1}$ matrix obtained by placing three copies of M side by side and B the $3 \times 3^{k+1}$ matrix determined by placing side by side the 3×3^k matrices B_1, B_2, B_3 where each column of B_1 is (the transpose of) (0, 1, 2), of B_2 is (1, 2, 0), and of B_3 is (2, 0, 1). Each of A and B has constant row sums and constant column sums.

Let $N = A + 3^{k+1}B$. Then N not only has constant row and column sums, but consists of the numbers $1, 2, \cdots, 3^{k+2}$ (why?). The row sums of M are each $(1/6)(3^{k+1})(3^{k+1}+1)$, so that the row sums of N are each

$$\begin{aligned} 3 \times (1/6)(3^{k+1})(3^{k+1}+1) + 3^{k+1}(3^k) + 3^{k+1}(2 \times 3^k) &= (1/6)[3^{k+2}(3^{k+1}+1)] + (3^{2k+1} \times 3) \\ &= (1/6)(3^{k+2})(3^{k+1} + 1 + 6 \times 3^k) = (1/6)(3^{k+2})(3^{k+2}+1) \ . \end{aligned}$$

The column sums of M are each $(3/2)(3^{k+1}+1)$ and so the column sums of N are each

$$(3/2)(3^{k+1}+1)+3^{k+1}+2\times 3^{k+1}=(1/2)(3^{k+2}+3+2\times 3^{k+2})=(3/2)(3^{k+2}+1)\ .$$

We now require that each of $(1/6)(3^{k+1})(3^{k+1}+1)$ and $(3/2)(3^{k+1}+1)$ be divisible by 6. This will occur exactly when $3^{k+1}+1\equiv 0 \pmod 4$, so that k must be even. Thus, we can obtain an array as desired when $n=9^m$ for some positive integer m. (Note that $9^m\equiv 9 \pmod{12}$.)

252. Suppose that a and b are the roots of the quadratic x^2+px+1 and that c and d are the roots of the quadratic x^2+qx+1. Determine $(a-c)(b-c)(a+d)(b+d)$ as a function of p and q.

Solution 1. From the theory of the quadratic, we have that $a+b=-p$, $c+d=-q$ and $ab=cd=1$. Then

$$\begin{aligned}(a-c)(b-c)(a+d)(b+d)&=(a-c)(b+d)(b-c)(a+d)\\&=(ab-cd+ad-bc)(ba-cd+bd-ca)\\&=(ad-bc)(bd-ca)=abd^2-a^2cd-b^2cd+abc^2\\&=d^2-a^2-b^2+c^2=[(c+d)^2-2cd]-[(a+b)^2-2ab]\\&=(q^2-2)-(p^2-2)=q^2-p^2\ .\end{aligned}$$

Solution 2. Using $a+b=-p$, $c+d=-q$ and $ab=cd=1$, we obtain that

$$\begin{aligned}(a-c)(b-c)(a+d)(b+d)&=[ab-(a+b)c+c^2][ab+(a+b)d+d^2]\\&=(1+pc+c^2)(1-pd+d^2)=(2+c^2+d^2)-p^2\\&=(c+d)^2-p^2=q^2-p^2\ .\end{aligned}$$

253. Let n be a positive integer and let $\theta=\pi/(2n+1)$. Prove that $\cot^2\theta$, $\cot^2 2\theta$, $\cdots$, $\cot^2 n\theta$ are the solutions of the equation

$$\binom{2n+1}{1}x^n-\binom{2n+1}{3}x^{n-1}+\binom{2n+1}{5}x^{n-2}-\cdots=0\ .$$

Solution 1. From de Moivre's Theorem that

$$\cos m\theta+i\sin m\theta=(\cos\theta+i\sin\theta)^m\ ,$$

we obtain from a comparison of imaginary parts that

$$\sin m\theta=\binom{m}{1}\cos^{m-1}\theta\sin\theta-\binom{m}{3}\cos^{m-3}\sin^3\theta+\cdots\ ,$$

for each positive integer m. Hence

$$\sin(2n+1)\theta=\sin^{2n+1}\theta\left[\binom{2n+1}{1}\cot^{2n}\theta-\binom{2n+1}{3}\cot^{2n-2}\theta+\cdots\right]\ .$$

When $\theta=(k\pi)/(2n+1)$ for $1\le k\le n$, $\sin(2n+1)\theta=0$ while $\sin\theta\neq 0$. The desired result follows.

Solution 2. [Y. Zhao] Observe that, for each complex a,

$$\frac{1}{2}[(a+1)^{2n+1}-(a-1)^{2n+1}]=\binom{2n+1}{1}a^{2n}+\binom{2n+1}{3}a^{2n-2}+\binom{2n+1}{5}a^{2n-4}+\cdots\ .$$

Suppose that $a = i\cot k\theta$ with $\theta = \pi/(2n+1)$ and $1 \leq k \leq n$. Note that $\sin k\theta \neq 0$. Then

$$\begin{aligned}\binom{2n+1}{1}(-\cot^2 k\theta)^n + \binom{2n+1}{3}(-\cot^2 k\theta)^{n-1} + \cdots &= \frac{1}{2}[(i\cot k\theta + 1)^{2n+1} - (i\cot\theta - 1)^{2n+1}] \\ &= \frac{1}{2}\left(\frac{i}{\sin k\theta}\right)^{2n+1}[(\cos k\theta - i\sin k\theta)^{2n+1} - (\cos k\theta + i\sin k\theta)^{2n+1}] \\ &= \left(\frac{i}{\sin k\theta}\right)^{2n+1}[-\sin(2n+1)k\theta] = \left(\frac{i}{\sin k\theta}\right)^{2n+1}[-\sin k\pi] = 0 \ ,\end{aligned}$$

and the result follows.

254. Determine the set of all triples (x, y, z) of integers with $1 \leq x, y, z \leq 1000$ for which $x^2 + y^2 + z^2$ is a multiple of xyz.

Solution. Suppose that $x^2 + y^2 + z^2 = kxyz$, for a positive integer k. It can be checked that if the equation is satisfied by (x, y, z) then it is also satisfied by $(x, y, kxy - z)$. Since $z^2 < x^2 + y^2 + z^2 = kxyz$, it follows that $z < kxy$, If $z > \frac{1}{2}kxy$, then $kxy - z < \frac{1}{2}kxy$. Suppose that we start with a solution. If we have, say z exceeding $\frac{1}{2}xy$, then we can replace z by a new value less than $\frac{1}{2}xy$. We can do the analogous thing with x and y. Every such operation reduces the sum $x + y + z$, so it can be performed at most finitely often, and we reach a situation where it cannot be done any more. Thus, we arrive at a solution where, say, $1 \leq x \leq y \leq z \leq kxy/2$, so that, in particular $kx \geq 2$. We can also start with such a solution and go backwards to achieve any given solution.

Since

$$x^2 + y^2 + \left(\frac{kxy}{2} - z\right)^2 = \left(\frac{kxy}{2}\right)^2 \ ,$$

it follows that

$$x^2 + y^2 + \left(\frac{kxy}{2} - y\right)^2 \geq \left(\frac{kxy}{2}\right)^2 \ ,$$

so that

$$3y^2 \geq x^2 + 2y^2 \geq kxy^2$$

and $kx \leq 3$. Thus $kx = 2$ or $kx = 3$.

The case $kx = 2$ leads to $x^2 + (y - z)^2 = 0$ which has no solutions as specified. Hence $kx = 3$ and $k = 1$ or $k = 3$. For these two cases, we find that the base solutions are respectively $(x, y, z) = (3, 3, 3)$ and $(x, y, z) = (1, 1, 1)$.

Suppose that $k = 1$. Modulo 3, any square is congruent to 0 or 1. If, say, $x \equiv 0 \pmod 3$, then $y^2 + z^2 \equiv 0 \pmod 3$; this can occur only if y and z are multiples of 3. Hence $(x, y, z) = (3u, 3v, 3v)$ for some integers u, v, w. But then $9u^2 + 9v^2 + 9w^2 = 27uvw$, or $u^2 + v^2 + w^2 = 3uvw$. Contrarily, any solution (u, v, w) of this equation gives rise to a solution (x, y, z) of $x^2 + y^2 + z^2 = xyz$. Therefore there is a one-one correspondence between solutions of $x^2 + y^2 + z^2 = xyz$ where all numbers are multiples of 3 and solutions of $x^2 + y^2 + z^2 = 3xyz$. We will obtain these solutions below.

The only other possibility is that none of x, y, z is divisible by 3. But then, $x^2 + y^2 + z^2$ would be a multiple of 3 and xyz not a multiple of 3; thus there are no solutions of this type.

Suppose that $k = 3$. From the above, we know that every solution arises from a solution for which $1 \leq x \leq y \leq z \leq 3xy/2$ and for such a solution $x = 1$. Let $(x, y, z) = (1, y, ty)$ where $1 \leq y \leq 3/2$. Then $1 + (1 + t^2)y^2 = 3ty^2$, so that

$$y^2 = \frac{1}{3t - 1 - t^2} = \frac{1}{\frac{5}{4} - (t - \frac{3}{2})^2} \ .$$

The denominator is not less than 1, so that $y^2 \leq 1$. Hence the only solution that can generate the rest is $(x, y, z) = (1, 1, 1)$.

To get a handle on the situation, fix $x = u$ and consider a sequence of solutions $(x, y, z) = (u, v_{n-1}, v_n)$. The solution $w = v_{n-1}$ satisfies the quadratic equation

$$u^2 + w^2 + v_n^2 = 3uwv_n$$

and so also does a second value $w = v_{n+1}$. By the theory of the quadratic, we have that

$$v_{n+1} + v_{n-1} = 3uv_n \tag{1}$$

and

$$v_{n+1}v_{n-1} = u^2 + v_n^2 \ . \tag{2}$$

If we start off with a solution $(x, y, z) = (u, v_0, v_1)$, we can use either (1) or (2) to determine the sequence $\{v_n\}$. Note that, since

$$v_{n+1} - v_n = v_n - v_{n-1} + (3u - 2)v_n > v_n - v_{n-1} \ ,$$

if $v_1 \geq v_0$, then the sequence $\{v_n\}$ is increasing. Note also, that the equations (1) and (2) are symmetric in v_{n-1} and v_{n+1}, so we can extend the sequence backwards as well as forwards.

Using the recursion $v_{n+1} = 3uv_n - v_{n-1}$, we get the following sequences for various values of u:

$$u = 1 \ : \quad \{v_n\} = \{1, 1, 2, 5, 13, 34, 89, 233, 610\}$$

$$u = 2 \ ; \quad \{v_n\} = \{1, 1, 5, 29, 169, 985\}$$

$$u = 5 \ ; \quad \{v_n\} = \{194, 13, 1, 2, 29, 433\}$$

$$u = 13 \ ; \quad \{v_n\} = \{34, 1, 5, 194\}$$

$$u = 29 \ ; \quad \{v_n\} = \{433, 5, 2, 169\}$$

$$u = 34 \ ; \quad \{v_n\} = \{13, 189\}$$

and so on. This yields the following solutions with $1 \leq x \leq y \leq z \leq 1000$: $(x, y, z) = (1, 1, 1)$, $(1, 1, 2)$, $(1, 5, 13)$, $(1, 13, 34)$, $(1, 34, 89)$, $(1, 89, 233)$, $(1, 233, 610)$, $(2, 5, 29)$, $(2, 29, 169)$, $(2, 169, 985)$, $(5, 13, 194)$, $(5, 29, 433)$.