Solutions

206. In a group consisting of five people, among any three people, there are two who know each other and two neither of whom knows the other. Prove that it is possible to seat the group around a circular table so that each adjacent pair knows each other.

Solution. Let the five people be A, B, C, D, E. We first show that each person must know exactly two of the others. Suppose, if possible, that A knows B, C, D. Then, by considering all the triples containing A, we see that each pair of B, C, D do not know each other, contrary to hypothesis. Thus, A knows at most two people. On the other hand, if A knows none of B, C and D, then each pair of B, C, D must know each other again yielding a contradiction. Therefore, A knows exactly two people, say B and E. Similarly, each of the others knows exactly two people.

Since A knows B and E, A does not know C and D, so, by considering the triple A, C, D, we see that C and D must know each other, and by considering the triple A, B, E, that B and E do not know each other. Thus, B knows A and one of C and D; suppose, say, that B knows C. Then B knows neither of D and E, so that D must know E. Hence, we can seat the people in the order $A--B--C--D--E$, and each adjacent pair knows each other.

207. Let n be a positive integer exceeding 1. Suppose that $A = (a_1, a_2, \cdots, a_m)$ is an ordered set of $m = 2^n$ numbers, each of which is equal to either 1 or -1. Let

$$S(A) = (a_1a_2, a_2a_3, \cdots, a_{m-1}a_m, a_ma_1) \ .$$

Define, $S^0(A) = A$, $S^1(A) = S(A)$, and for $k \geq 1$, $S^{k+1} = S(S^k(A))$. Is it always possible to find a positive integer r for which $S^r(A)$ consists entirely of 1s?

Solution 1. For $i > m = 2^n$, define $a_i = a_{i-m}$. Then, by induction, for positive integers r, we can show that the rth iterate of S acting on A is

$$S^r(A) = S(S^{r-1}(A)) = \left(\cdots, \prod_{i=0}^{r} a_{k+i}^{\binom{r}{i}}, \cdots \right) \ .$$

This is clear when $r = 1$. Suppose it holds for the index r. Then the kth term of $S^{r+1}(A)$ is equal to

$$\prod_{i=0}^{r} a_{k+i}^{\binom{r}{i}} \prod_{i=1}^{r+1} a_{k+i}^{\binom{r}{i-1}} = \prod_{i=0}^{r+1} a_{k+i}^{\binom{r+1}{i}} \ .$$

Now let $r = 2^n$. Then, for $1 \leq i \leq 2^{n-1}$,

$$\binom{2^n}{i} = \left(\frac{2^n}{i}\right)\left(\frac{2^n-1}{1}\right)\left(\frac{2^n-2}{2}\right)\cdots\left(\frac{2^n-i+1}{i-1}\right)$$

is even, since the highest power of 2 that divides $2^n - j$ is that same as the highest power of 2 that divides j for $1 \leq j \leq 2^n - 1$ and 2 divides i to a lower power than it divides 2^n. Hence the kth term of $S^m(A)$ is equal to $a_ka_{k+m} = a_k^2 = 1$, and so $S^m(A)$ has all its entries equal to 1.

Solution 2. [A. Chan] Defining a_i for all positive indices i as in the previous solution, we find that

$$S(A) = (a_1a_2, a_2a_3, a_3a_4, \cdots, a_ma_1)$$

$$S^2(A) = (a_1a_3, a_2a_4, a_3a_5, \cdots, a_ma_2)$$

$$S^4(A) = (a_1a_5, a_2a_6, a_3a_7, \cdots, a_ma_4)$$

$$S^8(A) = (a_1a_9, a_2a_{10}, \cdots, a_ma_8)$$

and so on, until we come to, for $m = 2^n$,

$$S^m(A) = (a_1a_{1+m}, a_2a_{2+m}, \cdots, a_ma_{2m}) = (a_1^2, a_2^2, \cdots, a_m^2) = (1, 1, \cdots, 1).$$

Solution 3. [R. Romanescu] We prove the result by induction on n. The result holds for $n = 1$, since for $A = (a_1, a_2)$, we have that $S(A) = (a_1a_2, a_2a_1)$, and $S^2(A) = (1, 1)$. Suppose, for vectors with 2^n entries, we have shown that $S^{2^n}(A) = (1, 1, \cdots, 1)$ for n-vectors A, for $n \geq 1$. Consider the following vector with 2^{n+1} entries: $A = (a_1, b_1, a_2, b_2, \cdots, a_m, b_m)$ where $m = 2^n$. Then

$$S^2(A) = (a_1a_2, b_1b_2, a_2a_3, b_2b_3, \cdots, a_{m-1}a_m, b_{m-1}b_m) \ ,$$

i.e., applying S twice is equivalent to applying S to the separate vectors consisting of the even entries and of the odd entries. Then, by the induction, applying S^2 2^n times (equivalent to applying S 2^{n+1} times), we get a vector consisting solely of 1s.

208. Determine all positive integers n for which $n = a^2 + b^2 + c^2 + d^2$, where $a < b < c < d$ and a, b, c, d are the four smallest positive divisors of n.

Solution. It is clear that $a = 1$. Suppose, if possible that n is odd; then its divisors a, b, c, d must be odd, and so $a^2 + b^2 + c^2 + d^2$ must be even, leading to a contradiction. Hence n must be even, and so $b = 2$, and exactly one of c and d is odd. Hence

$$n = a^2 + b^2 + c^2 + d^2 \equiv 1 + 0 + 1 + 0 = 2$$

mod 4, and so c must be an odd prime number and d its double. Thus, $n = 5(1 + c^2)$. Since c divides n, c must divide 5, and so $c = 5$. We conclude that $n = 130$.

209. Determine all positive integers n for which $2^n - 1$ is a multiple of 3 and $(2^n - 1)/3$ has a multiple of the form $4m^2 + 1$ for some integer m.

Solution. We first establish the following result: *let p be an odd prime and suppose that $x^2 \equiv -1$ (mod p) for some integer n; then $p \equiv 1$ (mod 4). Proof.* By Fermat's Little Theorem, $x^{p-1} \equiv 1$ (mod p), since x cannot be a multiple of p. Also $x^4 \equiv 1$ (mod p). Suppose that $p - 1 = 4q + r$ where $0 \leq r \leq 3$. Since $p - 1$ is even, so is r; thus, $r = 0$ or $r = 2$. Now $x^r \equiv x^rx^{4q} \equiv x^{p-1} \equiv 1$ (mod p), so $r = 0$. Therefore $p - 1$ is a multiple of 4. ♠

Suppose that 3 divides $2^n - 1$. Since $2^n \equiv (-1)^n$ (mod 3), n must be even. When $n = 2$, $(2^n - 1)/3 = 1$ has a multiple of the form $(2m)^2 + 1$; any value of m will do. Suppose that $n \geq 2$. Let $n = 2^u \cdot v$, with v odd and $u \geq 1$. Then

$$2^n - 1 = (2^v + 1)(2^v - 1)(2^w + 2^{w-2v} + \cdots + 2^{2v} + 1)$$

where $w = n - 2v = 2v(2^{u-1} - 1)$. Suppose that $(2m)^2 \equiv -1$ (mod $(2^n - 1)/3$)). Then, since $2^v + 1$ is divisible by 3, $(2m)^2 \equiv -1$ (mod $2^v - 1$), If $v \geq 3$, then $2^v - 1$ is divisible by a prime p congruent to 3 (mod 4) and, by the foregoing result, $x^2 \equiv -1$ (mod p) is not solvable. We are led to a contradiction, and so $v = 1$ and n must be a power of 2.

Now let $n = 2^u$. Then

$$2^n - 1 = (2 - 1)(2 + 1)(2^2 + 1)(2^4 + 1) \cdots (2^{2^{u-1}} + 1)$$

so that

$$\frac{2^n - 1}{3} = \prod_{i=1}^{u-1} (2^{2^i} + 1) \ .$$

We now use the *Chinese Remainder Theorem*: *if $q_1, q_2, \cdots, q_r$ are pairwise coprime integers and $a_i, a_2, \cdots, a_r$ arbitrary integers, then there exists an integer x such that $x \equiv a_i$ (mod $q_1q_2 \cdots q_r$) for $1 \leq i \leq r$, and x is unique up to a multiple of $q_1q_2 \cdots q_r$.* This is applied to $q_i = 2^{2^i} + 1$ $(1 \leq i \leq u-1)$ and $a_i = 2^{2^{i-1}-1}$. Observe that q_i and q_j are coprime for $i < j$. (For, if $2^{2^i} \equiv -1 \pmod p$, then $2^{2^j} \equiv 2^{2^{i+1}} \equiv 1 \pmod p$, so that $2^{2^j} + 1 \equiv 2 \pmod p$ and $p = 1$.) So there exists an integer m for which

$$m \equiv 2^{2^{i-1}-1} \pmod{2^{2^i}+1}$$

for $1 \leq i \leq u-1$. Therefore

$$4m^2 + 1 \equiv 2^2 \cdot 2^{2^i-2} + 1 \equiv 2^{2^i} + 1 \equiv 0$$

modulo $\prod_{i=1}^{u-1}(2^{2^i}+1)$ as desired.

For example, when $u = 3$, we have $m \equiv 1 \pmod 5$ and $m \equiv 2 \pmod{17}$, so we take $m = 36$ and find that $4m^2 + 1 = 61 \times 85 = 61 \times (\frac{1}{3} \times (2^8 - 1))$. When $u = 4$, we need to satisfy $m \equiv 1 \pmod 5$, $m \equiv 2 \pmod{17}$ and $m \equiv 8 \pmod{257}$: when $m = 3606$, $4m^2 + 1 = 52012045 = 2381 \times 5 \times 17 \times 257 = 2381 \times (\frac{1}{3} \times (2^{16} - 1))$.

210. ABC and DAC are two isosceles triangles for which B and D are on opposite sides of AC, $AB = AC$, $DA = DC$, $\angle BAC = 20°$ and $\angle ADC = 100°$. Prove that $AB = BC + CD$.

Solution 1. Produce BC to E so that $CE = CD$. Note that $\angle DCE = 60°$ (why?). Then ΔDCE is isosceles and so $\angle CDE = 60°$. Since $DA = DE$, we have that $\angle DAE = \angle DEA = 10°$. Therefore, $\angle BAE = 60° - 10° = 50°$ and $\angle BEA = 60° = 10° = 50°$, whence $AB = BE$.

Solution 2. Let $a = |AB| = |AC|$, $b = |BC|$, $c = |AD| = |CD|$, and $d = |BD|$. From the Law of Cosines applied to two triangles, we find that $d^2 = b^2 + c^2 + bc = a^2 + c^2 - ac$, whence $0 = b^2 - a^2 + (b+a)c = (b+a)(b-a+c)$. Therefore, $a = b + c$, as desired.

Solution 3. [M. Zaharia] From the Law of Sines, we have that $(\sin 80°)BC = (\sin 20°)AB$ and

$$(\sin 80°)CD = (\sin 100°)CD = (\sin 40°)AC = (\sin 40°)AB \ .$$

Hence

$$(\sin 80°)[BC + CD] = [\sin 20° + \sin 40°]AB = [2\sin 30° \cos 10°]AB \ .$$

Since $\sin 80° = \cos 10°$ and $\sin 30° = 1/2$, the result follows.

Solution 4. Since, in any triangle, longer sides are opposite larger angles, $AB = AC > AD$. Let E be a point of the side AB for which $AE = AD$. Then ΔAED is isosceles with apex angle $60°$, from which we find that $CD = AD = DE = AE$. Since ΔDEC is isosceles and $\angle EDC = \angle ADC - \angle ADE = 100° - 60° = 40°$, it follows that $\angle DEC = \angle DCE = 70°$, $\angle ACE = 70° - 40° = 30°$ and

$$\angle ECB = 80° - 30° = 50° = 120° - 70° = \angle DEB - \angle DEC = \angle CEB \ .$$

Hence $BE = BC$ and so $AB = AE + EB = CD + BC$.

Solution 5. Since $\angle ABC + \angle ADC = 80° + 100° = 180°$, $ABCD$ is a concyclic quadrilateral. Suppose, wolog, that the circumcircle has unit radius. Since AB, BC and CD subtend respective angles $160°$, $40°$, $80°$ at the centre of the circumcircle, $AB = 2\sin 80°$, $BC = 2\sin 20°$ and $CD = 2\sin 40°$. Since

$$\sin 20° + \sin 40° = 2\sin 30° \cos 10° = \sin 80° \ ,$$

the result follows.

211. Let ABC be a triangle and let M be an interior point. Prove that

$$\min\{MA, MB, MC\} + MA + MB + MC < AB + BC + CA \ .$$

Solution 1. Let D, E, F be the respective midpoints of BC, AC, AB. Suppose, wolog, M belongs to both of the trapezoids $ABDE$ and $BCEF$. Then

$$MA + MB < BD + DE + EA \quad \text{and} \quad MB + MC < BF + FE + EC$$

whence

$$MA + 2MB + MC < AB + BC + CA \ .$$

To see, for example, that $MA+MB < BD+DE+EA$, construct GH such that G lies on the segment BD, H lies on the segment AE, $GH \| DE$ and M lies on the segment GH. Then

$$\begin{aligned} AM + MB &< AH + HM + MG + GB = AH + HG + GB \\ &< AH + HD + DG + GB = AH + HD + DB \\ &< AH + HE + ED + DB = EA + DE + BD \ . \end{aligned}$$

Solution 2. [R. Romanescu] We first establish that, if W is an interior point of a triangle XYZ, then $XW + WY < XZ + ZY$. To see this, produce YW to meet XZ at V. Then

$$XW + YW < XV + VW + YW = XV + VY < XV + VZ + ZY = XZ + ZY \ .$$

Let AP, BQ, CR be the medians of triangle ABC. These medians meet at the centroid G and partition the triangle into six regions. Wolog, suppose that M is in the triangle AGR. Then $AM + MB < AG + GB$ and $AM + MC < AR + RC$. Hence $2AM + MB + MC < AG + GB + AR + RC$. Since $AP < AR + RP = \frac{1}{2}(AB + AC)$, $AG = \frac{2}{3}AP < \frac{1}{3}(AB + BC)$. Similarly, $BG < \frac{1}{3}(AB + AC)$. Also $CR < \frac{1}{2}(AC + BC)$ and $AR = \frac{1}{2}AB$. Hence

$$\begin{aligned} AG + GB + AR + RC &< \frac{7}{6}AB + \frac{5}{6}AC + \frac{5}{6}BC \\ &< AB + \frac{1}{6}(AC + BC) + \frac{5}{6}AC + \frac{5}{6}BC \\ &= AB + BC + CA \ . \end{aligned}$$

The result now follows.

212. A set S of points in space has at least three elements and satisfies the condition that, for any two distinct points A and B in S, the right bisecting plane of the segment AB is a plane of symmetry for S. Determine all possible finite sets S that satisfy the condition.

Solution. We first show that all points of S lie on the surface of a single sphere. Let U be the smallest sphere containing all the points of S. Then there is a point $A \in S$ on the surface of U. Let B be any other point of S and P be the right bisecting plane of the segment AB. Since this is a plane of symmetry for S, the image V of the sphere U reflected in P must contain all the points of S. Let W be the sphere whose equatorial plane is $P \cap U = P \cap V$. Then $S \subseteq U \cap V \subseteq W \subseteq U \cup V$. Since U is the smallest sphere containing S and W is symmetric about P, $U \subseteq W$, $V \subseteq W$ and $U \cap V = U \cup V$. Hence $U = V$ and P must be an equatorial plane of U. But this means that B must lie on the surface of U.

Consider the case that S is a planar set; then the points of S lie on a circle. Let three of them in order be A, B, C. Since the image of B reflected in the right bisector of AC is a point of S on the arc AC, it can only be B itself. Hence $AB = BC$. Since S is finite, S must consist of the vertices of a regular polygon.

In general, any plane that intersects S must intersect it in the vertices of a regular polygon, so that, in particular, all the faces of the convex hull of S are regular polygons. Let F be one of these faces and G and H be faces adjacent to F sharing the respective edges AB and BC with F. Then G and H are images of each other under the reflection in the right bisector of AC, and so must be congruent. Consider the vertex

B of F; if I is a face adjacent to G and contains the vertex B, then F and I must be congruent. In this way, we can see that around each vertex of the convex hull of S, every second face is congruent. Thus, the polyhedron has all its faces of one or two types of congruent regular polygons. Since every vertex can be carried into every other by a sequence of reflections in right bisectors of edges, each vertex must have the same number of faces that contain it.

Since all the angles of faces meeting at a given vertex must sum to less than $360°$ and since all the faces are regular polygons, there must be 3, 4 or 5 faces at each vertex. If all the faces are congruent, the convex hull must be a regular polyhedron whenever S has at least four points. If S consists of the vertices of a regular tetrahedron or a regular octahedron, the conditions of the problem are satisfied. Otherside, it is possible to find an edge and a vertex whose plane intersects the polyhedron in a non-equilateral triangle so S cannot be at the vertices of a cube, a regular dodecahedron or a regular icosahedron.

If the polyhedron has two types of faces, then at each vertex, there must be two equilateral triangles and either two squares or two pentagons. Suppose that PQR is one of the triangle faces, and that T is the other end of the edge emanating from R. Then the plane PQT cuts the polyhedron in the non-equilateral triangle PQT (note that all sides have the same length, so there are no other points of S on this plane). Hence, this possibility must be rejected.