199. Let A and B be two points on a parabola with vertex V such that VA is perpendicular to VB and θ is the angle between the chord VA and the axis of the parabola. Prove that

$$\frac{|VA|}{|VB|} = \cot^3 \theta \ .$$

Comment. A lot of students worked harder on this problem than was necessary. It should be noted that all parabolas are similar (as indeed all circles are similar); this means that you can establish a general result about parabolas by dealing with a convenient one. Let us see why this is so. One definition of a parabola is that it is the locus of points that are equidistant from a given point (called the *focus*) and a given line (called the *directrix*) that does not contain the point. Any point-line pair can be used, and each such pair can be transformed into another by a similarity transformation. (Translate one point on to the other, make a rotation to make the two lines parallel and perform a dilation about the point that makes the two lines coincide.) The same transformation will take the prabola defined by one pair to the parabola defined by the other. You should point out in your solution that there is no loss of generality in taking the particular case of a parabola whose equation in the plane is $y = ax^2$. But you do not have to be even that general; it is enough to assume that the parabola has the equation $y = x^2$ or $x = y^2$. (*Exercise:* Determine the focus and the directrix for these parabolas.) Some of the solvers did not appear to be aware that parabolas need not have vertical or horizontal axes; the axis of a parabola can point in any direction.

Solution. Wolog, suppose that the parabola is given by $y^2 = x$, so that its vertex is the origin and its axis is the $x-$axis. Suppose $A \sim (u, v)$ is a point on the parabola whose radius vector makes an angle θ with the axis; then $v/u = \tan\theta$. Hence $1/u = v^2/u^2 = \tan^2\theta$, so that $A \sim (\cot^2\theta, \cot\theta)$. Similarly, it can be shown that $B \sim (\tan^2\theta, -\tan\theta)$. Hence

$$\frac{|VA|^2}{|VB|^2} = \frac{\cot^2\theta(\cot^2\theta + 1)}{\tan^2\theta(\tan^2\theta + 1)} = \cot^6\theta \ ,$$

and the result follows.

200. Let n be a positive integer exceeding 1. Determine the number of permutations $(a_1, a_2, \cdots, a_n)$ of $(1, 2, \cdots, n)$ for which there exists exactly one index i with $1 \leq i \leq n-1$ and $a_i > a_{i+1}$.

Comment. Some solvers found it difficult to appreciate what was going on in this problem. It is often a good beginning strategy to actually write out the appropriate permutations for low values of n. This does two things for you. First, it gives you a sense of what goes into constructing the right permutations and so how your argument can be framed. Secondly, it gives you some data against which you can check your final answer.

Solution 1. For $n \geq 1$, let p_n be the number of permutations of the first n natural numbers that satisfy the condition. Suppose that $a_i = n$ for some i with $1 \leq i \leq n-1$. Then $(a_1, a_2, \cdots, a_{i-1})$ and $(a_{i+1}, \cdots, a_n)$ must both be in increasing order, so that the appropriate permutation is determined uniquely once its first $i-1$ entries are found. There are $\binom{n-1}{i-1}$ ways of choosing these entries. If $a_n = n$, then there are p_{n-1} ways of ordering the first $n-1$ numbers to give an appropriate permutation. Hence

$$p_n = \left[\sum_{i=1}^{n-1} \binom{n-1}{i-1}\right] + p_{n-1} = 2^{n-1} - 1 + p_{n-1} \ .$$

Thus, substituting for each p_i in turn, we have that

$$p_n = (2^{n-1} - 1) + (2^{n-2} - 1) + \cdots + (2^2 - 1) + (2 - 1) + (1 - 1) = 2^n - 1 - n = 2^n - (n+1) \ .$$

Solution 2. [H. Li; M. Zaharia] For $n \geq 2$, let p_n be the number of acceptable permutations. We have that $p_2 = 1$. Consider first the placing of the numbers $1, 2, \cdots, n-1$ in some order. If they appear in their natural order, then we can slip in n before any one of them to get an acceptable permutation; there are $n-1$ ways of doing this. If there exists a single consecutive pair (r, s) of numbers for which $r < s$ and r follows s, then we can slip n between s and r or at the end to get an acceptable permutation. There are $2p_{n-1}$ possibilities. If there is more than one pair (r, s) of consecutive pairs with $r < s$ and r following s, then no placement of n will yield an acceptable permutation. Hence

$$p_n = 2p_{n-1} + (n-1)$$

so that

$$\begin{aligned} p_n + n + 1 &= 2(p_{n-1} + n) = 2^2(p_{n-2} + n - 1) \\ &= \cdots = 2^{n-2}(p_2 + 3) = 2^{n-2} \cdot 4 = 2^n \ , \end{aligned}$$

whence $p_n = 2^n - (n+1)$.

Solution 3. [R. Barrington Leigh] Let $1 \leq k \leq n-1$ and let (x, y) be a pair of integers for which $1 \leq y < x \leq n$ and $x - y = k$. There are $n - k$ such pairs, $(1, k+1), (2, k+2), \cdots, (n-k, k)$. For each such pair, we consider suitable permutations for which x and y are adjacent in the order (x, y). Then the numbers $1, 2, \cdots, y-1$ must precede and $x+1, \cdots, n$ must follow the pair. The remaining $k-1$ numers from $x+1$ to $x+k-1 = y-1$ can go either before or after the pair; there are 2^{k-1} possibilities. Once it is decided whether each of these goes before or after the pair, there is only one possible arrangement. Hence the number of permutations of the required type is

$$\begin{aligned} \sum_{k=1}^{n-1} (n-k)2^{k-1} &= \sum_{k=1}^{n-1} [(n-k+1)2^k - (n-k+2)2^{k-1}] \\ &= \sum_{k=1}^{n-1} [(n - \overline{k-1})2^k - (n - \overline{k-2})2^{k-1} \\ &= 2 \cdot 2^{n-1} - (n+1) = 2^n - (n+1) \ . \end{aligned}$$

Solution 4. Let $1 \leq i \leq n-1$ and consider the number of suitable permutations for which $a_i > a_{i+1}$. There are $\binom{n}{i}$ possible choices of $\{a_1, a_2, \cdots, a_i\}$ with $a_1 < a_2 < \cdots < a_i$, and except for the single case of $\{1, 2, \cdots, i\}$, the maximum element a_i of each of them exceeds the minimum element a_{i+1} of its complement $\{a_{i+1}, \cdots, a_n\}$. Hence the number of permutations is

$$\sum_{i=1}^{n-1} \left[\binom{n}{i} - 1 \right] = \sum_{i=0}^{n} \left[\binom{n}{i} - 1 \right] = 2^n - (n+1) \ .$$

Solution 5. (Variant of Solution 4.) We can form an acceptable permutation in the following way. Let $1 \leq k \leq n$. Select any subset of k numbers in one of $\binom{n}{k}$ ways and place them in ascending order at the beginning of the arrangement and place the other $n-k$ at the end, again in ascending order. This fails to work only when the set chosen is $\{1, 2, \cdots, k\}$. Hence the total number of ways is

$$\sum_{k=1}^{n} \left[\binom{n}{k} - 1 \right] = \left[\sum_{k=1}^{n} \binom{n}{k} \right] - n = (2^n - 1) - n \ .$$

201. Let $(a_1, a_2, \cdots, a_n)$ be an arithmetic progression and $(b_1, b_2, \cdots, b_n)$ be a geometric progression, each of n positive real numbers, for which $a_1 = b_1$ and $a_n = b_n$. Prove that

$$a_1 + a_2 + \cdots + a_n \geq b_1 + b_2 + \cdots + b_n \ .$$

Solution 1. The result is obvious if $a_1 = a_n = b_1 = b_n$, as then all of the a_i and b_j are equal. Suppose that the progressions are nontrivial and that the common ratio of the geometric progression is $r \neq 1$. Observe that

$$(r^{n-1} + 1) - (r^{n-k} + r^{k-1}) = (r^{k-1} - 1)(r^{n-k} - 1) > 0 \ .$$

Then

$$\begin{aligned}
b_1 + b_2 + \cdots + b_n &= b_1(1 + r + r^2 + r^3 + \cdots + r^{n-1}) \\
&= \frac{b_1}{2}\sum_{k=1}^{n}(r^{n-k} + r^{k-1}) \\
&< \frac{b_1 n}{2}(r^{n-1} + 1) = \frac{n}{2}[b_1 r^{n-1} + b_1] \\
&= \frac{n}{2}[b_n + b_1] = \frac{n}{2}[a_n + a_1] = a_1 + a_2 + \cdots + a_n \ .
\end{aligned}$$

Solution 2. For $1 \leq r \leq n$, we have that

$$\begin{aligned}
b_r &= b_1^{(n-r)/(n-1)} b_n^{(r-1)/(n-1)} \\
&= a_1^{(n-r)/(n-1)} a_n^{(r-1)/(n-1)} \\
&\leq \frac{n-r}{n-1}a_1 + \frac{r-1}{n-1}a_n = a_r \ ,
\end{aligned}$$

by the arithmetic-geometric means inequality.

202. For each positive integer k, let $a_k = 1 + (1/2) + (1/3) + \cdots + (1/k)$. Prove that, for each positive integer n,

$$3a_1 + 5a_2 + 7a_3 + \cdots + (2n+1)a_n = (n+1)^2 a_n - \frac{1}{2}n(n+1) \ .$$

Solution 1. Observe that, for $1 \leq k \leq n$,

$$(2k+1) + (2k+3) + \cdots + (2n+1) = (1 + 3 + \cdots + \overline{2n+1}) - (1 + 3 + \cdots + \overline{2k-1}) = (n+1)^2 - k^2 \ .$$

Then

$$\begin{aligned}
&3a_1 + 5a_2 + 7a_3 + \cdots + (2n+1)a_n \\
&\qquad = (3 + 5 + \cdots + \overline{2n+1}) \cdot 1 + (5 + 7 + \cdots + \overline{2n+1}) \cdot \left(\frac{1}{2}\right) + \cdots + (2n+1)\left(\frac{1}{n}\right) \\
&\qquad = \sum_{k=1}^{n}[(n+1)^2 - k^2]\left(\frac{1}{k}\right) = (n+1)\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n} k \\
&\qquad = (n+1)^2 a_n - \frac{1}{2}n(n+1) \ .
\end{aligned}$$

Solution 2. Observe that for each positive integer $k \geq 2$,

$$\begin{aligned}
&[(k+1)^2 a_k - \frac{1}{2}k(k+1)] - [k^2 a_{k-1} - \frac{1}{2}(k-1)k] \\
&\qquad = k^2(a_k - a_{k-1}) + (2k+1)a_k - \frac{1}{2}k(k + 1 - \overline{k-1}) \\
&\qquad = k^2(1/k) + (2k+1)a_k - k = (2k+1)a_k \ .
\end{aligned}$$

Hence

$$\begin{aligned}
3a_1 + 5a_2 + &\cdots + (2n+1)a_n \\
&= 3a_1 + \sum_{k=2}^{n} \{[(k+1)^2 a_k - \frac{1}{2}k(k+1)] - [k^2 a_{k-1} - \frac{1}{2}(k-1)k]\} \\
&= 3a_1 + [(n+1)^2 a_n - \frac{1}{2}n(n+1)] - [4a_1 - 1] \\
&= (n+1)^2 a_n - \frac{1}{2}n(n+1) + 1 - a_1 = (n+1)^2 a_n - \frac{1}{2}n(n+1) \ .
\end{aligned}$$

Solution 3. We use an induction argument. The result holds for $k = 1$. Suppose it holds for $n = k-1 \geq 1$. Then

$$\begin{aligned}
3a_1 + 5a_2 + &\cdots + (2k-1)a_{k-1} + (2k+1)a_k \\
&= k^2 a_{k-1} - \frac{1}{2}k(k-1) + (2k+1)a_k \\
&= k^2 \left(a_k - \frac{1}{k} \right) - \frac{1}{2}k(k-1) + (2k+1)a_k \\
&= (k+1)^2 a_k - [k + \frac{1}{2}k(k-1)] \\
&= (k+1)^2 a_k - \frac{1}{2}k(k+1) \ .
\end{aligned}$$

Solution 4. [R. Furmaniak] Let $a_0 = 0$, Then $a_i = a_{i-1} + (1/i)$ for $1 \leq i \leq n$, so that

$$\begin{aligned}
\sum_{i=1}^{n} (2i+1)a_i &= \sum_{i=1}^{n} [(i+1)^2 - i^2] a_i \\
&= \sum_{i=1}^{n} [(i+1)^2 a_i - i^2 a_{i-1} - i^2(1/i)] \\
&= (n+1)^2 a_n - a_0 - \sum_{i=1}^{n} i = (n+1)^2 a_n - \frac{1}{2}n(n+1) \ .
\end{aligned}$$

Solution 5. [A. Verroken] Let $a_0 = 0$. For $n \geq 1$,

$$\begin{aligned}
(n+1)^2 a_n &= \sum_{k=0}^{n} (2k+1) a_n \\
&= \sum_{k=0}^{n} (2k+1) \left[a_k + \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{n} \right) \right] \\
&= \sum_{k=1}^{n} (2k+1) a_k + \sum_{k=0}^{n-1} (2k+1) \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{n} \right) \\
&= \sum_{k=1}^{n} (2k+1) a_k + \sum_{k=0}^{n-1} \left(\frac{1}{k+1} \right) (1 + 3 + \cdots + (2k+1)) \\
&= \sum_{k=1}^{n} (2k+1) a_k + \sum_{k=0}^{n-1} \left(\frac{1}{k+1} \right) (k+1)^2
\end{aligned}$$

from which the result follows. (To see the second last equality, write out the sums and instead of summing along the $2k+1$, sum along the $1/(k+1)$.]

Solution 6. [T. Yin] Recall Abel's Partial Summation Formula:

$$\sum_{k=1}^{n} u_k v_k = (u_1 + u_2 + \cdots + u_n)v_n - \sum_{k=1}^{n-1}(u_1 + u_2 + \cdots + u_k)(v_{k+1} - v_k) \ .$$

(Prove this. Compare with integration by parts in calculus.) Applying this to $u_k = 2k + 1$ and $v_k = a_k$, we find that $u_1 + \cdots + u_k = (k + 1)^2 - 1$ and $v_{k+1} - v_k = 1/(k + 1)$, whereupon

$$\begin{aligned}\sum_{k=1}^{n}(2k+1)a_k &= (n+1)^2 a_n - a_n - \sum_{k=1}^{n-1}(k+1) + \sum_{k=1}^{n-1}\frac{1}{k+1} \\ &= (n+1)^2 a_n - a_n - \left[\frac{n(n+1)}{2} - 1\right] + [a_n - 1] \\ &= (n+1)^2 a_n - \frac{n(n+1)}{2} \ .\end{aligned}$$

203. Every midpoint of an edge of a tetrahedron is contained in a plane that is perpendicular to the opposite edge. Prove that these six planes intersect in a point that is symmetric to the centre of the circumsphere of the tetrahedron with respect to its centroid.

Solution 1. Let O be the centre of the circumsphere of the tetrahedron $ABCD$ and G be its centroid. Then

$$\overrightarrow{OG} = \frac{1}{4}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}) \ .$$

Let N be the point determined by

$$\overrightarrow{ON} = 2\overrightarrow{OG} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}) \ .$$

Let P be the midpoint of the edge AB. Then

$$\overrightarrow{PN} = \overrightarrow{ON} - \overrightarrow{OP} = \overrightarrow{ON} - \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) = \frac{1}{2}(\overrightarrow{OC} + \overrightarrow{OD})$$

and

$$\overrightarrow{PN} \cdot \overrightarrow{CD} = \frac{1}{2}(\overrightarrow{OD} + \overrightarrow{OC}) \cdot (\overrightarrow{OD} + \overrightarrow{OC}) = \frac{1}{2}(|\overrightarrow{OD}|^2 - |\overrightarrow{OC}|^2) = 0 \ .$$

Hence $\overrightarrow{PN} \perp \overrightarrow{CD}$, so that the segment PN is contained in a plane that is orthogonal to CD. A similar result holds for the other five edges. The result follows.

Solution 2. [O. Bormashenko] Let O be the circumcentre and let G be the centroid of the tetrahedron. Let M be the midpoint of the edge AB and N the midpoint of the edge CD. The centroid of the triangle ABC lies at a point E on MC for which $CE = 2EM$, so that $CM = 3EM$. The centroid of the tetrahedron is the position of the centre of gravity when unit masses are placed at its vertices, and so is the position of the centre of gravity of a unit mass placed at D and a triple mass at E. Thus G is on DE and satisfies $DG = 3GE$.

Consider triangle CDE. We have that

$$\frac{CM}{ME} \cdot \frac{EG}{GD} \cdot \frac{DN}{NC} = (-3) \cdot \left(\frac{1}{3}\right) \cdot 1 = -1 \ ,$$

so that, by the converse to Menelaus' Theorem, G, M and N are collinear. Consider triangle MCN and transversal DGE. By Menelaus' Theorem,

$$-1 = \frac{ME}{EC} \cdot \frac{CD}{DN} \cdot \frac{NG}{GM} = \left(\frac{1}{2}\right) \cdot (-2) \cdot \frac{NG}{GM} \ ,$$

whence $NG = GM$ and G is the midpoint of MN.

Suppose that K is the point on OG produced so that $OG = GK$. Since OK and MN intersect in G at their respective midpoints, $OMKN$ is a planar parallelogram and $ON \| KM$. Since $OC = OD$, triangle OCD is isosceles, and so $ON \perp CD$. Hence $KM \perp CD$. Therefore, K lies on the plane through the midpoint M of AB and perpendicular to CD. By symmetry, K lies on the other planes through the midpoints of an edge and perpendicular to the opposite edge.

204. Each of $n \geq 2$ people in a certain village has at least one of eight different names. No two people have exactly the same set of names. For an arbitrary set of k names (with $1 \leq k \leq 7$), the number of people containing at least one of the k names among his/her set of names is even. Determine the value of n.

Solution 1. Let P be a person with the least number of names. The remaining $n-1$ people have at least one of the names not possessed by P, so by the condition of the problem applied to the set of names not possessed by P, $n-1$ is even and so n is odd. Let x be one of the eight names, and suppose, if possible, that no person has x as his/her sole name. Then all n people have at least one of the remaining names which yields the contradiction that n must be even. Hence, for each name, there is a person with only that name. Suppose there is no person with only a pair $\{x, y\}$ of names. Then there are $n-2$ people who have a name other than x and y, which yields again a contradiction, since $n-2$ is odd. Hence, for each pair of names, there is exactly one person possessing those two names.

We can continue the argument. Suppose, if possible, there is no person possessing exactly the three names x, y and z. Then except for the six people with the name sets $\{x\}$, $\{y\}$, $\{z\}$, $\{x, y\}$, $\{y, z\}$, $\{z, x\}$, everyone possesses at least one of the names other than x, y, z, which leads to a contradiction. Eventually, we can argue that, for each nonvoid set of the eight names, there is exactly one person with that set of names. Since there are $255 = 2^8 - 1$ such subsets, there must be 255 people.

Solution 2. [R. Furmaniak] For $1 \leq i \leq 8$, let S_i be the set of people whose names include the ith name. By the condition of the problem for $k = 1$, the cardinality, $\#S_i$, of S_i must be even. Suppose, for $2 \leq k \leq 7$, it has been shown that any intersection of fewer than k of the S_i has even cardinality.

Consider an intersection of k of the S_i, say $S_1 \cap S_2 \cap \cdots \cap S_k$. By the condition of the problem, $\#(S_1 \cup S_2 \cup \cdots \cup S_k)$, the number of people with at least one of the first k names, is even. But, from the Principle of Inclusion-Exclusion, we have that

$$\#(S_1 \cup S_2 \cup \cdots \cup S_k) = \sum_{i=1}^{k} \#S_i - \sum_{i \neq j} \#(S_i \cap S_j) + \sum_{i,j,k} \#(S_i \cap S_j \cap S_k) - \cdots + (-1)^k \#(S_1 \cap S_2 \cap \cdots \cap S_k) \ .$$

By the induction hypothesis, each term in the series on the right but the last is even, and so the last is even as well.

Consider the largest set of names, say $\{i_1, \cdots, i_r\}$ possessed by any one person. This set can appear only once, so that $\cap_{j=1}^{r} S_{i_j}$ is a singleton. By the above paragraph, the intersection must have eight members (no fewer) and so some person possesses all eight names.

If a set of names does not belong to any person, let T be a maximal such set with $k \leq 7$ names, say the first k names. By maximality, each superset of T be be a set of names for someone. The supersets consist of the k names along with all of the $2^{8-k} - 1$ possible subsets of the remaining names. But the superset of names are possessed by all the people in $S_1 \cap S_2 \cap \cdots \cap S_k$, and this set has even cardinality and so cannot have cardinality $2^{8-k} - 1$. This is a contradiction. Thus every possible nonvoid set of names must occur and $n = 2^8 - 1$.

205. Let $f(x)$ be a convex realvalued function defined on the reals, $n \geq 2$ and $x_1 < x_2 < \cdots < x_n$. Prove that

$$x_1 f(x_2) + x_2 f(x_3) + \cdots + x_n f(x_1) \geq x_2 f(x_1) + x_3 f(x_2) + \cdots + x_1 f(x_n) \ .$$

Solution 1. The case $n = 2$ is obvious. For $n = 3$, we have that

$$\begin{aligned} x_1 f(x_2) + & x_2 f(x_3) + x_3 f(x_1) - x_2 f(x_1) - x_3 f(x_2) - x_1 f(x_3) \\ &= (x_3 - x_2) f(x_1) + (x_2 - x_1) f(x_3) - (x_3 - x_1) f(x_2) \\ &= (x_3 - x_1) \left[\frac{(x_3 - x_2)}{(x_3 - x_1)} f(x_1) + \frac{(x_2 - x_1)}{(x_3 - x_1)} f(x_3) - f(x_2) \right] \geq 0 \ . \end{aligned}$$

Suppose, as an induction hypothesis, that the result holds for all values of n up to $k \geq 3$. Then

$$\begin{aligned} x_1 f(x_2) + x_2 f(x_3) + \cdots + x_k f(x_{k+1}) + x_{k+1} f(x_k) \\ &= [x_1 f(x_2) + \cdots + x_k f(x_1)] + [x_k f(x_{k+1}) + x_{k+1} f(x_1) - x_k f(x_1)] \\ &\geq [x_2 f(x_1) + \cdots + x_1 f(x_k)] + [x_{k+1} f(x_k) + x_1 f(x_{k+1}) - x_1 f(x_k)] \\ &= x_2 f(x_1) + \cdots + x_{k+1} f(x_k) + x_1 f(x_{k+1}) \ , \end{aligned}$$

by the result for $n = k$ and $n = 3$.

Solution 2. [J. Kramar] For $1 \leq i \leq n$, let $\lambda_i = (x_i - x_1)/(x_n - x_1)$, so that $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n = 1$ and $x_i = \lambda_i x_n + (1 - \lambda_i) x_1$. Then

$$\begin{aligned} f(x_n)\lambda_{n-1} + (1 - \lambda_2) f(x_1) &= (f(x_n) - f(x_1))\lambda_{n-1} + f(x_1)(\lambda_{n-1} + \lambda_n - \lambda_2) \\ &= (f(x_n) - f(x_1))(\lambda_{n-1}\lambda_n - \lambda_1\lambda_2) + f(x_1)(\lambda_n + \lambda_{n-1} - \lambda_2 - \lambda_1) \\ &= (f(x_n) - f(x_1)) \sum_{i=2}^{n-1} (\lambda_i \lambda_{i+1} - \lambda_{i-1}\lambda_i) + f(x_1) \sum_{i=2}^{n-1} (\lambda_{i+1} - \lambda_{i-1}) \\ &= \sum_{i=2}^{n-1} [\lambda_{i+1} - \lambda_{i-1}][\lambda_i f(x_n) + (1 - \lambda_i) f(x_1)] \\ &\geq \sum_{i=2}^{n-1} [\lambda_{i+1} - \lambda_{i-1}] f(x_i) \ . \end{aligned}$$

Multiplying by $x_n - x_1$ and rearranging terms yields that

$$x_{n-1} f(x_n) + x_n f(x_1) \geq \left[\sum_{i=2}^{n-1} f(x_i)(x_{i+1} - x_{i-1}) \right] + x_1 f(x_n) + x_2 f(x_1)$$

from which the desired result follows.