Solutions

262. Let ABC be an acute triangle. Suppose that P and U are points on the side BC so that P lies between B and U, that Q and V are points on the side CA so that Q lies between C and V, and that R and W are points on the side AB so that R lies between A and W. Suppose also that

$$\angle APU = \angle AUP = \angle BQV = \angle BVQ = \angle CRW = \angle CWR \ .$$

The lines AP, BQ and CR bound a triangle T_1 and the lines AU, BV and CW bound a triangle T_2. Prove that all six vertices of the triangles T_1 and T_2 lie on a common circle.

Solution 1. Note that the configuration requires the feet of the altitudes to be on the interior of the sides of the triangle and the orthocentre to be within the triangle. Let θ be the common angle referred to in the problem. Let XYZ be that triangle with sides parallel to the sides of triangle ABC and A on YZ, B on ZX and C on XY. Then A, B, C are the respective midpoints of YZ, ZX, XY and the orthocentre H or triangle ABC is the circumcentre of triangle XYZ. [Why?] Let ρ be the common length of HX, HY, HZ.

Let K be the intersection of AP and BQ (a vertex of T_1). Since $\angle KPC + \angle KQC = 180°$, $CQKP$ is concyclic. Hence $\angle AKB + \angle AZB = \angle PKQ + \angle PCQ = 180°$ and $AKBZ$ is concyclic. Since $AH \perp BC$ and $BH \perp AC$, the angle between AH and BH is equal to $\angle ACB = \angle XZY$, so that $AHBZ$ is also concyclic. Thus, A, H, K, B, Z lie on a common circle, so that $\angle HKZ = \angle HBC = 90°$.

Now $\theta = \angle APC = \angle ZAK = \angle ZHK$, so that, in the right triangle HKZ, $|HK| = \rho \cos\theta$. Similarly, it can be shown that the distance from each vertex of triangle T_1 and T_2 from H is $\rho\cos\theta$ and the result follows.

Solution 2. [R. Dan] Let H be the orthocentre of triangle ABC, let AP and BQ intersect at D, and let AU and BQ intersect at E. Triangle APU is isosceles with $AP = AU$, and AH a bisector of $\angle PAU$ and a right bisector of PU. Suppose $X = AH \cap BC$, $Y = BH \cap AC$ and $Z = CH \cap AB$.

Since triangles APU and BQV are similar, $\angle PAH = \angle QBH$, so that $ABDH$ is concyclic and $\angle ADH = \angle ABH$. Similarly, $ACEH$ is concyclic and $\angle AEH = \angle ACH$. Since the quadrilateral $BZYC$ has right angles at Z and Y, $BZYC$ is concyclic and

$$\angle ABH = \angle ABY = \angle ZBY = \angle ZCY = \angle ZCA = \angle ACH \ .$$

Therefore, $\angle ADH = \angle ABH = \angle ACH = \angle AEH$.

Since AH is common, $\angle ADH = \angle AEH$ and $\angle DAH = \angle EAH$, triangles ADH and AEH are congruent (ASA) and $HD = HE$. Thus, H is equidistant from the intersections of BQ with both AP and AU. Similarly, H is equidistant from the intersection of AP and both BQ and BV. Following around, we can show that H is equidistant from all the vertices of triangle T_1 and T_2, and the result follows.

263. The ten digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are each used exactly once altogether to form three positive integers for which the largest is the sum of the other two. What are the largest and the smallest possible values of the sum?

Solution 1. Since the sum has at least as many digits as either of the summands, the sum must have at least four digits. However, the number of digits of the sum cannot exceed one more than the number of digits of the larger summand. Hence, the sum can have at most five digits. However, a five-digit sum must arise from the sum of a four-digit number which is at most 9876 and a single-digit number which is at most 9. Since this means that the sum cannot exceed 9885, we see that a five-digit sum is impossible.

A four-digit sum can arise either as the sum of two three-digit numbers or as the sum of a four-digit and a two-digit number. In the former case, the sum must exceed 1000 and be less than 2000 and, in the latter case, it must be at least 2000.

Thus, the smallest possible sum must be obtained by adding two three-digit numbers to get a four-digit sum. Since the digits of the sum are all distinct, the smallest possible sum is at least 1023. Since $589 + 437 = 1026$, the smallest sum is at most 1026. We may assume that each digit in the first summand exceeds the corresponding digit in the second summand. The only possibilities for a lower sum are

$$5pq + 4rs = 1023 \ , \quad 6pq + 3rs = 1024 \ , \quad 6pq + 3rs = 1025 \ ,$$

for digits p, q, r, s. One can check that none of these works.

For the largest sum, let the first summand have four digits and the second two. The hundreds digit of the first summand is 9 and the thousands digit of the sum exceeds the thousands digit of the first summand by 1. Since $5987 + 34 = 6021$, the largest sum is at least 6021. The only possibilities to consider for a larger sum are

$$79ab + cd = 80ef \ , \quad 69ab + cd = 70ef \ , \quad 59ab + cd = 60ef \ ,$$

for digits a, b, c, d, e, f. It can be checked that none of these works.

Thus, the smallest sum is 1026 and the largest is 6021.

Solution 2. [C. Shen] As in Solution 1, we eliminate the possibility of a five-digit sum. Suppose that we have

$$a9bc + de = f0gh$$

with digits a, b, c, d, e, f, g, h and $f = a + 1$. There must be a carry from adding the tens digits and we have two possibilities:

$$c + e = h \ , \quad b + d = 10 + g \ ; \tag{1}$$

$$c + e = 10 + h \ , \quad b + d = 9 + g \ . \tag{2}$$

In case (1), we have that

$$\begin{aligned} 36 &= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = a + b + c + d + e + f + g + h \\ &= a + 10 + 2g + 2h + f = 2(5 + g + h + a) + 1 \ , \end{aligned}$$

which is impossible, as the two sides have different parities. In case (2), we have that

$$36 = a + 9 + 2g + 10 + 2h + f = 2(10 + a + g + h) \ ,$$

so that $a + g + h = 8$. Since a, g, h are all positive integers, $a \leq 5$ and we have the case $59bc + de = 60gh$ with $c + e \geq 11$. The only possibilities for (c, e) are $(8, 3)$, $(8, 4)$, $(7, 4)$, and these lead to

$$5978 + 43 = 6021 \ , \quad 5978 + 34 = 6012 \ , \quad 5987 + 34 = 6021 \ .$$

The largest sum is 6021.

The smallest sum is at least 1023 and at most $1026 = 589 + 437$. Suppose that

$$pqr + uvw = 102x$$

with $3 \leq x \leq 6$. Since $r + w \geq 3 + 4 = 7$ and $q + v \geq 7$, we have

$$r + w = 10 + x \ , \quad q + v = 11 \ , \quad p + u = 9 \ .$$

Hence

$$\begin{aligned} 42 &= 3 + 4 + 5 + 6 + 7 + 8 + 9 = p + q + r + u + v + w + x \\ &= 9 + 11 + 2x + 10 = 30 + 2x \ , \end{aligned}$$

so that $x = 6$ and 1026 is the smallest sum.

264. For the real parameter a, solve for real x the equation

$$x = \sqrt{a + \sqrt{a+x}} \ .$$

A complete answer will discuss the circumstances under which a solution is feasible.

Solution 1. Suppose that $y = \sqrt{a+x}$. Note that x and y are both nonnegative. Then $x^2 - a = y$ and $y^2 - a = x$, whence

$$0 = (x^2 - y^2) + (x - y) = (x-y)(x+y+1) \ .$$

Since $x + y + 1 \geq 1$, it follows that $y = x$ and so

$$0 = x^2 - x - a = (x - (1/2))^2 - ((1/4) + a) \ .$$

For a real solution, we require that $a \geq -1/4$. For $-1/4 \leq a \leq 0$, both the sum and the product of the solutions are nonnegative and we get the candidates

$$x = \frac{1 \pm \sqrt{1+4a}}{2} \ .$$

When $a > 0$, the equation has a positive and a negative solution, and only the positive solution

$$x = \frac{1 + \sqrt{1+4a}}{2}$$

is up for consideration.

We check that these solutions work. When $a \geq -1/4$, $x = \frac{1}{2}(1 + \sqrt{1+4a})$ and

$$\begin{aligned} a + x &= \frac{2a + 1 + \sqrt{1+4a}}{2} = \frac{4a + 2 + 2\sqrt{4a+1}}{4} \\ &= \left(\frac{1 + \sqrt{4a+1}}{2}\right)^2 , \end{aligned}$$

so that

$$a + \sqrt{a+x} = \frac{2a + 1 + \sqrt{1+4a}}{2} = \left(\frac{1+\sqrt{4a+1}}{2}\right)^2 = x^2 \ .$$

When $-1/4 \leq a \leq 0$, $x = \frac{1}{2}(1 - \sqrt{1+4a})$,

$$a + x = \frac{2a + 1 - \sqrt{1+4a}}{2} = \left(\frac{1 - \sqrt{4a+1}}{2}\right)^2 \ .$$

so that

$$a + \sqrt{a+x} = a + \left(\frac{1 - \sqrt{4a+1}}{2}\right) = \left(\frac{1-\sqrt{4a+1}}{2}\right)^2 = x^2 \ .$$

(Note that, when $a > 0$,

$$\sqrt{a+x} = \frac{\sqrt{4a+1} - 1}{2}$$

and we get an extraneous solution.)

Solution 2. $x = \sqrt{a + \sqrt{a+x}} \Longrightarrow x^2 - a = \sqrt{a+x}$

$$\Longrightarrow 0 = x^4 - 2ax^2 - x + a^2 - a = (x^2 - x - a)(x^2 + x - a + 1) \ .$$

We analyze the possibilities from $x^2 - x - a = 0$ as in Solution 1. If, on the other hand, $x^2 + x - (a-1) = 0$, then $x = \frac{1}{2}(-1 \pm \sqrt{4a-3})$, which is real when $a \geq 3/4$. The possibility $x = \frac{1}{2}(\sqrt{4a-3} - 1)$ leads to

$$x + a = \left(\frac{\sqrt{4a-3}+1}{2}\right)^2$$

and

$$a + \sqrt{a+x} = \frac{2a+1+\sqrt{4a-3}}{2} \neq \left(\frac{\sqrt{4a-3}-1}{2}\right)^2 .$$

Thus, $x = \frac{1}{2}(\sqrt{4a-3} - 1)$ is extraneous. Since $\frac{1}{2}(-1 - \sqrt{4a-3}) < 0$, $x = \frac{1}{2}(-1 - \sqrt{4a-3}) < 0$ is also extraneous,

Solution 3. For a solution, we require that $x \geq 0$. By squaring twice, we are led to the equation

$$0 = x^4 - 2ax^2 - x + a^2 - a = a^2 - (2x^2+1)a + (x^4 - x) .$$

Solving for a yields

$$\begin{aligned} a &= \frac{(2x^2+1) + \sqrt{(2x^2+1)^2 - 4(x^4-x)}}{2} = \frac{(2x^2+1) + \sqrt{4x^2+4x+1}}{2} \\ &= \frac{(2x^2+1)+(2x+1)}{2} = x^2 + x + 1 , \end{aligned}$$

or

$$a = \frac{(2x^2+1)-(2x+1)}{2} = x^2 - x .$$

(Note that the proper square root has been extracted since $x \geq -1/2$.) In the first case

$$\sqrt{a+\sqrt{a+x}} = \sqrt{x^2+x+1+\sqrt{x^2+2x+1}} = \sqrt{x^2+2x+2}$$

¿ x .

In the second case,

$$\sqrt{a+\sqrt{a+x}} = \sqrt{x^2 - x + \sqrt{x^2}} = \sqrt{x^2} = x .$$

Thus, only the case, $a = x^2 - x$ leads to a valid solution. Note that $a = x^2 - x = (x - \frac{1}{2})^2 - \frac{1}{4}$, so that $a \geq -\frac{1}{4}$ for a solution to work. Since we require $x \geq 0$ and $a = x(x-1)$, we see from the graph of this equation that there are two valid values of x when $-\frac{1}{4} \leq a \leq 0$ and one valid value of x when $0 < a$.

Solution 4. [J. Zhao] For any real x, one of the following must hold:

$$x > \sqrt{a+x} ; \qquad x < \sqrt{a+x} ; \qquad x = \sqrt{a+x} .$$

In case of the first,

$$x > \sqrt{a+x} > \sqrt{a+\sqrt{a+x}} \neq x ,$$

so that such x does not satisfy the equation. Similarly, we can reject any x satisfying the second condition as a solution of the equation. Hence for every solution of the given equation, we must have that $x = \sqrt{a+x}$ or $x^2 - x - a = 0$. We now finish off as in the previous solutions.

Comment. Many solvers did not pay attention to the feasibility of the solutions. Solution 3 was particularly insidious, because it was easy to skip the analysis that only one of the values of x gave a solution when $a > 0$. Surd equations is a dandy topic for students to lose points they should gain because of carelessness or a superficial treatment.

265. Note that $959^2 = 919681$, $919 + 681 = 40^2$; $960^2 = 921600$, $921 + 600 = 39^2$; and $961^2 = 923521$, $923 + 521 = 38^2$. Establish a general result of which these are special instances.

Solution. Let $b \geq 2$ be a base of enumeration. Then we wish to investigate solutions of the system

$$(b^k - u)^2 = b^k v + w \tag{1}$$

$$(u - 1)^2 = v + w \tag{2}$$

where k, u, v are positive integers and the integer w satisfies $0 \leq w \leq b^k - 1$. The numerical examples given correspond to $(b, k, u) = (10, 3, 41), (10, 3, 40)$ and $(10, 3, 39)$. Subtracting (2) from (1) yields

$$(b^{2k} - 1) - 2(b^k - 1)u = (b^k - 1)v$$

whence $v = (b^k + 1) - 2u$ and $w = u^2 - b^k$. We require that $b^k \leq u^2 \leq 2b^k - 1$ in order to get a generalization. So, to generate examples of the phenomenon, first select a base b and a parameter k for the number of digits; then select u to satisfy the foregoing inequality. Then one can check, with v and w determined, the desired system of equations holds.

Consider first the situation $b = 10$. When $k = 1$, we have that $u = 4$ and we get the case $6^2 = 36, 3+6 = 3^2$. When $k = 2$, we have that $10 \leq u \leq 14$ and find that $86^2 = 7396, 73+96 = 13^2$ and so on up to $90^2 = 8100, 81+0 = 9^2$. When $k = 3$, we have that $32 \leq u \leq 44$, and find that $956^2 = 913936, 913+936 = 43^2$ and so on up to $968^2 = 937024, 937 + 24 = 31^2$.

Examples from base 3 are $5^2 = (221)_3$, $(2 + 21)_3 = 3^2$; $6^2 = (1100)_3$, $(11 + 0)_3 = 2^2$; $20^2 = (112211)_3$, $(112 + 211)_3 = 6^2$; $21^2 = (121100)_3$, $(121 + 100)_3 = 5^2$.

Comment. In the above system, we could replace $u - 1$ by $u - d$ and get other instances. For example, with $(b, k) = (10, 2)$, we can get the instances $(27^2 = 729, 7 + 29 = 6^2)$, $(29^2 = 841, 8 + 41 = 7^2)$, $(30^2, 3^2)$, $(39^2, 6^2)$, $(40^2, 4^2)$, $(50^2, 5^2)$, $(57^2, 9^2)$. $(60^2, 6^2)$, $(70^2, 7^2)$, $(75^2, 9^2)$, $(78^2, 12^2)$, $(80^2, 8^2)$ and $(98^2, 10^2)$.

Another formulation is to note that the numerical equations are special instances of the system $n^2 = b^k x + y$; $(n - x)^2 = x + y$ with $0 \leq y \leq b^k - 1$ and $0 \leq n < b^k$, where $(n, b, x, y) = (959, 10, 919, 681)$, $(960, 10, 921, 600)$, $(961, 10, 923, 521)$. These equations imply that $x(2n - x) = (b^k - 1)x$, whence $x = 2n - (b^k - 1)$. Thus,

$$n^2 = 2b^k n - b^{2k} + b^k + y \Longrightarrow y = (b^k - n)^2 - b^k \ ,$$

and we require that $b^k \leq (b^k - n)^2 < 2b^k - 1$. The analysis can be continued from here.

This question was, on the whole, badly done. In describing the generalization, one needs to provide a road map whereby one can make the appropriate subsitutions to obtain further examples. Many solvers were content to write down some equations of which the numerical examples were an instance without analyzing the conditions under which the equations could be used to obtain further examples. In effect, no further information was provided to show where other examples might be found.

266. Prove that, for any positive integer n, $\binom{2n}{n}$ divides the least common multiple of the numbers $1, 2, 3, \cdots, 2n - 1, 2n$.

Solution. We first establish that

$$0 \leq \lfloor 2x \rfloor - 2\lfloor x \rfloor \leq 1$$

for each positive real x. ($\lfloor \cdot \rfloor$ refers to "the greatest integer not exceeding".) If, for some integer s, $2s \leq 2x \leq 2s + 1$, then $s \leq x < s + \frac{1}{2}$ and $\lfloor 2x \rfloor = 2\lfloor x \rfloor$; if $2s + 1 \leq 2x < 2s + 2$, then $s + \frac{1}{2} \leq x < s + 1$ and $\lfloor 2x \rfloor = 2s + 1 = 2\lfloor x \rfloor + 1$.

Let p be a prime divisor of $\binom{2n}{n}$, so that $p \leq 2n$. Suppose that p^k is the highest power of p that divides an integer not exceeding $2n$. Then $p^k \leq 2n < p^{k+1}$. The exponent of p in the prime factorization of $\binom{2n}{n}$ is equal to

$$\left(\left\lfloor\frac{2n}{p}\right\rfloor+\left\lfloor\frac{2n}{p^2}\right\rfloor+\left\lfloor\frac{2n}{p^3}\right\rfloor+\cdots+\left\lfloor\frac{2n}{p^k}\right\rfloor\right)-2\left(\left\lfloor\frac{n}{p}\right\rfloor+\left\lfloor\frac{n}{p^2}\right\rfloor+\cdots+\left\lfloor\frac{n}{p^k}\right\rfloor\right)$$
$$=\sum_{i=1}^{k}\left(\left\lfloor\frac{2n}{p^i}\right\rfloor-2\left\lfloor\frac{n}{p^i}\right\rfloor\right)\le k\ .$$

Hence the exponent of p in the prime factorization of $\binom{2n}{n}$ does not exceed the exponent of p in the prime factorization of the least common multiple of the first $2n$ positive integers, for each prime divisor of $\binom{2n}{n}$. The result follows.

267. A non-orthogonal reflection in an axis a takes each point on a to itself, and each point P not on a to a point P' on the other side of a in such a way that a intersects PP' at its midpoint and PP' always makes a fixed angle θ with a. Does this transformation preserves lines? preserve angles? Discuss the image of a circle under such a transformation.

Solution. We suppose that $\theta \neq 90°$. The transformation preserves lines. This is clear for any line parallel to a. Let AB be a line through A that meets a at P, and let A' and B' be the reflective images of A and B. Since $AA' \| BB'$ and a is a median from P of triangles PAA' and PBB', it follows that A', B' and P are colllinear. Thus, any point on the line AP gets carried to a point on the line $A'P$. However, angles are not preserved. A line perpendicular to a is carried to a line making an angle not equal to a right angle with a (while a is kept fixed). [What is the angle of intersection?]

Suppose that the axis of reflection is the y axis. Let A and B be mutual images with A to the left and B to the right of the axis, AB meeting the axis at P and the upper right (and lower left) angles of intersection being θ. If $A \sim (x, y)$ (with $x \le 0$), then $P \sim (0, y - x\cot\theta)$ and $B \sim (-x, y - 2x\cot\theta)$. If $B \sim (u, v)$ (with $u \ge 0$), then $P \sim (0, v - u\cot\theta)$ and $A \sim (-u, v - 2u\cot\theta)$. Thus the transformation is given by

$$(x, y) \longrightarrow (X, Y) \equiv (-x, y - 2x\cot\theta)\ .$$

Consider the particular case of the circle with equation $x^2 + y^2 = 1$. The image curve has equation

$$1 = (-X)^2 + (Y - 2X\cot\theta)^2 = (1 + 4\cot^2\theta)X^2 - 4XY\cot\theta + Y^2$$

and this does not represent a circle. Thus, circles are not preserved under the transformation. In fact, a general circle with equation of the form $(x - a)^2 + (y - b)^2 = r^2$ gets carried to an second degree curve in the plane which turns out to be an ellipse.

Comment. A synthetic way of analyzing the image of a circle is to note that two chords of a circle that bisect each other must be diameters, and so have the same length. Using this, one can argue that a circle with one diameter along the axis and other perpendicular to the axis does not go to a circle.

268. Determine all continuous real functions f of a real variable for which

$$f(x + 2f(y)) = f(x) + y + f(y)$$

for all real x and y.

Solution 1. First, we show that $f(u) = 0$ if and only if $u = 0$. Suppose that $f(u) = 0$. Then, for all x,

$$f(x) = f(x + 2f(u)) = f(x) + u + f(u) = f(x) + u$$

so that $u = 0$. On the other hand, let $v = f(0)$. Taking $(x, y) = (-2v, 0)$ in the condition yields that

$$v = f(0) = f(-2v + 2v) = f(-2v) + 0 + v\ ,$$

whence

$$0 = f(-2v) = f(-2v + 2f(-2v)) = 0 - 2v + 0 = -2v$$

and $v = 0$.

Setting $y = x$ yields

$$f(x + 2f(x)) = x + 2f(x)$$

for all x. Let $g(x) \equiv x + 2f(x)$. Then $f(g(x)) = g(x)$ so that

$$\frac{1}{2}[g(g(x)) - g(x)] = g(x)$$

whence

$$g(g(x)) = 3g(x) \ .$$

Note also that $g(0) = 0 + 2f(0) = 0$. If $g(x) \equiv 0$, then $f(x) = -x/2$, and this is a valid solution. Suppose that $g(z) = a \neq 0$ for some z and a. Then, as $g(0) = 0$ and g is continuous, g assumes all values between 0 and a (by the intermediate value theorem). But $g(a) = g(g(z)) = 3g(z) = 3a$, so by the same argument, g assumes all values between 0 and $3a$. We can continue on to argue that g assumes all values between 0 and $3^k a$ for each positive integer k. Thus g assumes all positive values if $a > 0$ and assumes all negative values if $a < 0$.

Suppose that the former holds. Then, for all $x \geq 0$, we have that $g(x) = 3x$ and so $x + 2f(x) = 3x$, whence $f(x) = x$. Therefore, when x is arbitrary and $y \geq 0$, $f(x + 2y) = f(x) + 2y$. In particular, $0 = f(-2y + 2y) = f(-2y) + 2y$ so that $f(-2y) = -2y$. Hence, for all x, we must have that $f(x) = x$. A similar argument can be followed to show that $f(x) \equiv x$ when $a < 0$. Therefore, the only two solutions are $f(x) = x$ and $f(x) = -x/2$.

Solution 2. [S. Eastwood] Setting $x = y$, we find that $f(x + 2f(x)) = x + 2f(x)$ for all real x. Let

$$A = \{x + f(x) : x \in \mathbf{R}\} \ .$$

Then A is a nonvoid set. Suppose that $a \in A$. Then $f(a) = a$, so that $3a = a + 2f(a) \in A$. Hence $a \in A \Rightarrow 3a \in A$. Since $x \to x + 2f(x)$ is continuous, it satisfies the intermediate value theorem and so A must be of one of the following types: $A = \{0\}$; $A = [b, \infty)$, $A = [-b, \infty)$, $A = \mathbf{R}$ for some nonnegative value of b.

Suppose that $A = \{0\}$. Then, for all real x, $x + 2f(x) = 0$ and so $f(x) = -x/2$. This is a valid solution.

Suppose that A has a nonzero element a. Then

$$a = f(a) = f(-a + 2a) = f(-a + 2f(a)) = f(-a) + a + f(a) = f(-a) + 2a \ ,$$

whence $f(-a) = -a$ and $-3a = -a + 2f(-a) \in A$. Hence A must contain numbers that are both positive and negative, and so must consist of the whole set of reals. Hence, for all real x, $f(x) = x$, and this also is valid.

Solution 3. [J. Zhao] Suppose that $f(u) = f(v)$. Then, for each x,

$$f(x) + u + f(u) = f(x + f(u)) = f(x + f(v)) = f(x) + v + f(v)$$

so that $u = v$. Hence, f is one-one. Since f is continuous, f is always strictly increasing or always strictly decreasing on $\mathbf{R}$.

Suppose that f is increasing. Then

$$\lim_{x \to +\infty} x + 2f(x) = +\infty$$

and

$$\lim_{x\to-\infty} x + 2f(x) = -\infty \ ,$$

so that $x + f(x)$ assumes every real value (by the intermediate value theorem). Suppose that z is any real number. Select y such that $y + 2f(y) = z$. Then

$$f(z) = f(y + 2f(y)) = f(y) + y + f(y) = z \ ,$$

and so $f(x) \equiv x$. This works.

Suppose that f is decreasing. Suppose, if possible, that p and q are such that $p + 2f(p) < q + 2f(q)$. Then $f(p + 2f(p)) > f(q + 2f(q))$. But, we get a contradiction since $f(p + 2f(p)) = p + 2f(p)$ and $f(q + 2f(q)) = q + 2f(q)$. Hence, there is a constant c such that, for all real x, $x + 2f(x) = c$. Hence, $f(x) = -\frac{1}{2}(-x + c)$. Plugging this into the functional equation, we find that $c = 0$, and so we obtain the solution $f(x) = -x/2$.

Comments. If we assume that $f(x)$ is a polynomial, then it can be shown that its degree must be 1. Assuming a solution $f(x) = cx + d$ for constants c and d leads to the equations $d = 0 = 2c^2 - c - 1 = (2c + 1)(c - 1)$. Thus, it is not hard to get a partial solution.

There were a number of approaches to ascertaining that $f(0) = 0$. A. Critch began with the observation that $f(2f(0)) = f(0 + 2f(0)) = f(0) + 0 + f(0) = 2f(0)$. Let $a = 2f(0)$, so that $f(a) = a$. Furthermore,

$$f(2a) = f(a + a) = f(a + 2f(0)) = a + f(0) = (3a/2)$$

and

$$f(2a) = f(0 + 2a) = f(0 + 2f(a)) = f(0) + a + f(a) = (5a/2) \ .$$

This leads to $a = 0$.

R. Dan noted that $f(2f(y)) = y + f(y)$, and then went on to derive

$$f(-2f(y) + 2f(y)) = f(-2f(y)) + y + f(y) = f(-2f(y)) + f(2f(y)) \ .$$

Along with the property that $f(0) = 0$, one can then show that f assumes both positive and negative values.