PROBLEMS FOR SEPTEMBER 2004

Please mail your solutions to
Professor E.J. Barbeau
Department of Mathematics
University of Toronto
Toronto, ON M5S 3G3
no later than **November 15, 2004**. Please make sure that your name, full mailing address and email address appear on the front page of your solutions. If you do not write your family name last, please indicate it with an asterisk (*).

Notes. A sequence $x_1, x_2, \cdots, x_k$ is in *arithmetic progression* iff $x_{i+1} - x_i$ is constant for $1 \leq i \leq k-1$. A *triangular number* is a positive integer of the form

$$T(x) \equiv \frac{1}{2}x(x+1) = 1 + 2 + \cdots + x \ ,$$

where x is a positive integer.

332. What is the minimum number of points that can be found (a) in the plane, (b) in space, such that each point in, respectively, (a) the plane, (b) space, must be at an irrational distance from at least one of them?

333. Suppose that a, b, c are the sides of triangle ABC and that a^2, b^2, c^2 are in arithmetic progression.

(a) Prove that $\cot A$, $\cot B$, $\cot C$ are also in arithmetic progression.

(b) Find an example of such a triangle where a, b, c are integers.

334. The vertices of a tetrahedron lie on the surface of a sphere of radius 2. The length of five of the edges of the tetrahedron is 3. Determine the length of the sixth edge.

335. Does the equation
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{12}{a+b+c}$$
have infinitely many solutions in positive integers a, b, c?

336. Let $ABCD$ be a parallelogram with centre O. Points M and N are the respective midpoints of BO and CD. Prove that the triangles ABC and AMN are similar if and only if $ABCD$ is a square.

337. Let a, b, c be three real numbers for which $0 \leq c \leq b \leq a \leq 1$ and let w be a complex root of the polynomial $z^3 + az^2 + bz + c$. Must $|w| \leq 1$?

338. A *triangular triple* (a, b, c) is a set of three positive integers for which $T(a) + T(b) = T(c)$. Determine the smallest triangular number of the form $a + b + c$ where (a, b, c) is a triangular triple. (*Optional investigations:* Are there infinitely many such triangular numbers $a + b + c$? Is it possible for the three numbers of a triangular triple to each be triangular?)