Solutions

283. (a) Determine all quadruples (a, b, c, d) of positive integers for which the greatest common divisor of its elements is 1,

$$\frac{a}{b} = \frac{c}{d}$$

and $a + b + c = d$.

(b) Of those quadruples found in (a), which also satisfy

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{a} \text{ ?}$$

(c) For quadruples (a, b, c, d) of positive integers, do the conditions $a+b+c = d$ and $(1/b)+(1/c)+(1/d) = (1/a)$ together imply that $a/b = c/d$?

Solution 1. (a) Suppose that the conditions on a, b, c, d are satisfied. Note that b and c have symmetric roles. Since $ad = bc$, if b and c were both even, then either a or d would be even, whence both would be even (since $a + b + c = d$), contradicting the fact that the greatest common divisor of a, b, c, d is equal to 1. Hence, at most one of b and c is even.

Suppose, if possible, b and c were both odd. Then a and d would be odd as well. If $b \equiv c \pmod 4$, then $bc \equiv 1$ and $b + c \equiv 2 \pmod 4$, whence $ad \equiv a(a + 2) \equiv 3 \not\equiv bc \pmod 4$. If $b \equiv c + 2 \pmod 4$, it can similarly be shown that $ad \not\equiv bc \pmod 4$, In either case, we get an untenable conclusion. Hence, exactly one of b and c is even and the other is odd.

Without loss of generality, we may suppose that a and b have opposite parity. Let g be the greatest common divisor of a and b, so that $a = gu$ and $b = gv$ for some coprime pair (u, v) of positive integers with opposite parity. Since $d > c$, it follows that $b > a$ and $v > u$. Let $w = v - u$.

Since

$$\frac{b}{a} = \frac{a+b+c}{c} = \frac{a+b}{c} + 1 \ ,$$

it follows that

$$\frac{b-a}{a(b+a)} = \frac{1}{c}$$

whence

$$c = \frac{gu(u+v)}{w} \qquad \text{and} \qquad d = \frac{gv(u+v)}{w} \ .$$

Since the greatest common divisor of u and v is 1, w has no positive divisor in common with either u or v, save 1. Any common divisor of w and $u + v$ must divide $2u = (u + v) - (v - u)$ and $2v = (u + v) + (v - u)$; such a common divisor equals 1. Since u and v have opposite parity and so w is odd, w must divide g. Since the greatest common divisor of a, b, c, d is equal to 1, we must have that $g = w$. Hence

$$(a, b, c, d) = (u(v - u), v(v - u), u(v + u), v(v + u))$$

where u and v are coprime with opposite parity. Interchanging, the roles of b and c leads also to

$$(a, b, c, d) = (u(v - u), u(v + u), v(v - u), v(v + u))$$

with u, v coprime of opposite parity. On the other hand, any quadruples of this type satisfy the condition.

(b)

$$\begin{aligned}\frac{1}{b} + \frac{1}{c} + \frac{1}{d} &= \frac{1}{v(v-u)} + \frac{1}{u(v+u)} + \frac{1}{v(v+u)} \\ &= \frac{1}{v(v-u)} + \frac{1}{uv} = \frac{u+(v-u)}{uv(v-u)} = \frac{1}{v-u} = \frac{1}{a} \ .\end{aligned}$$

(c) Note that the conditions imply that $d-a$ and $b+c$ are nonzero. The conditions yield that $d-a=b+c$ and $(1/a)-(1/d)=(1/b)+(1/c)$. The second of these can be rewritten

$$\frac{ad}{d-a}=\frac{bc}{b+c}$$

so that $ad=bc$. Thus, all quadruples imply the required condition.

Solution 2. (a) [M. Lipnowski] Let $a/b=c/d=r/s$ where the greatest common divisor of r and s is equal to 1. Then $a=hr$, $b=hs$, $c=kr$, $d=ks$. Since the greatest common divisor of a,b,c,d equals 1, the greatest common divisor of h and k is 1. From $a+b+c=d$, we have that $(h+k)r=(k-h)s$. Observe that $\gcd(h+k,k-h)=1$ when h and k have opposite parity and $\gcd(h+k,k-h)=2$ when h and k are both odd. (Why?)

Thus, when h and k have oppposite parity, $r=k-h$, $s=k+h$ and

$$(a,b,c,d)=(h(k-h),h(k+h),k(k-h),k(k+h))$$

and, when h and k are both odd, then $r=\frac{1}{2}(k-h)$, $s=\frac{1}{2}(k+h)$ and

$$(a,b,c,d)=((1/2)h(k-h),(1/2)h(k+h),(1/2)k(k-h),(1/2)k(k+h))\ .$$

It can be checked that these always work. (Collate these with the result given in Solution 1.)

(b) Since $a/b=c/(a+b+c)$, $c=a(a+b)/(b-a)$ and $d=(a+b)+[a(a+b)/(b-a)]=b(a+b)/(b-a)$. Hence

$$\begin{aligned}\frac{1}{b}+\frac{1}{c}+\frac{1}{d}&=\frac{1}{b}+\frac{b-a}{a+b}\left(\frac{1}{a}+\frac{1}{b}\right)\\&=\frac{1}{b}+\frac{b-a}{ab}=\frac{1}{a}\ .\end{aligned}$$

(c) [M. Lipnowski]

$$\frac{1}{b}+\frac{1}{c}+\frac{1}{a+b+c}=\frac{1}{a}$$

is equivalent to

$$\begin{aligned}0&=bc(a+b+c)-a(b+c)(a+b+c)-abc\\&=(b+c)(bc-a^2-ab-ac)\ ,\end{aligned}$$

which in turn is equivalent to

$$0=bc-a^2-ab-ac\Longleftrightarrow bc=a(a+b+c)=ad\ .$$

284. Suppose that $ABCDEF$ is a convex hexagon for which $\angle A+\angle C+\angle E=360°$ and

$$\frac{AB}{BC}\cdot\frac{CD}{DE}\cdot\frac{EF}{FA}=1\ .$$

Prove that

$$\frac{AB}{BF}\cdot\frac{FD}{DE}\cdot\frac{EC}{CA}=1\ .$$

Solution 1. [A. Zhang] Since the hexagon is convex, all its angles are less than 180°. A dilation of factor $|CD|/|DE|$ followed by a rotation, both with centre D, takes E to C and F to a point G so that $\Delta DCG\sim\Delta DEF$, $\angle DEF=\angle DCG$ and $DE:EF:FD=DC:CG:GD$. Since $DE:DC=FD:GD$ and $\angle EDC=\angle FDG$, $\Delta EDC\sim\Delta FDC$ and $DE:DC:CE=FD:DG:GF$. Now

$$\angle DCG+\angle BCD=\angle DEF+\angle BCD=360°-\angle FAB>180°$$

so that C lies within the triangle BDG and $\angle BCG = 360° - (\angle DCG + \angle BCD) = \angle FAB$.

Also,

$$\frac{CG}{CD} = \frac{EF}{DE} = \frac{AF}{AB} \cdot \frac{BC}{CD}$$

so that $CG : BC = AF : AB$, with the result that $\Delta BCG \sim \Delta BAF$, $AB : BF : FA = CB : BG : GC$ and $\angle FBG = \angle ABC$. From the equality of these angles and $AB : CB = BF : BG$, we have that $\Delta ABC \sim \Delta FBG$ and $AB : BC : CA = FB : BG : GF$. Hence

$$\frac{AB}{BF} \cdot \frac{FD}{DE} \cdot \frac{EC}{CA} = \frac{CA}{GF} \cdot \frac{GF}{CE} \cdot \frac{CE}{CA} = 1$$

as desired.

Solution 2. [T. Yin] *Lemma.* Let $ABCD$ be a convex quadrilateral with a, b, c, d, p, q the respective lengths of AB, BC, CD, DA, AC and BD. Then

$$p^2q^2 = (ac + bd)^2 - 4abcd \cos^2 \theta$$

where $2\theta = \angle A + \angle C$.

Proof of Lemma. Locate E within the quadrilateral so that $\angle EDC = \angle ADB$ and $\angle ECD = \angle ABD$. Then $\Delta ABD \sim \Delta ECD$ whence $ac = qx$ where x is the length of EC. Now $\angle ADE = \angle BDC$ and $AD : DE = BD : CD$ whence $\Delta ADE \sim \Delta BDC$ and $bd = qy$ with y the length of AE.

Hence $abcd = q^2xy$ and $ac + bd = q(x + y)$. Therefore,

$$a^2c^2 + b^2d^2 + 2abcd = q^2(x^2 + 2xy + y^2) = q^2(x^2 + y^2) + 2abcd$$

which reduces to $a^2c^2 + b^2d^2 = q^2(x^2 + y^2)$.

Since $\angle DEC = \angle BAD$ and $\angle AED = \angle BCD$,

$$\angle AEC = \angle AED + \angle DEC = \angle C + \angle A = 2\theta \ .$$

By the law of cosines,

$$p^2 = x^2 + y^2 - 2xy \cos 2\theta = x^2 + y^2 - 2xy(2cos^2\theta - 1) \Longrightarrow$$

$$\begin{aligned} a^2c^2 + b^2d^2 &= p^2q^2 + 4q^2xy \cos^2 \theta - 2q^2xy \\ &= p^2q^2 + 4abcd \cos^2 \theta - 2abcd \end{aligned}$$

so that the desired result follows. ♠

Note that, when $\angle A + \angle C = 180°$, then we get Ptolemy's Theorem. Consider the hexagon of the problem with $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DE| = d$, $|EF| = e$, $|FA| = f$, $|BF| = g$, $|CA| = h$, $|CF| = m$, $|DF| = u$ and $|CE| = v$. We are given that $ace = bdf$ and need to prove that $auv = dgh$.

From the lemma applied to $ABDF$, we obtain that

$$g^2h^2 = a^2m^2 + 2abfm + b^2f^2 - 4abfm \cos^2 \alpha$$

where $2\alpha = \angle BAC + \angle BCF$. Applying the lemma to $CDEF$ yields that

$$u^2v^2 = d^2m^2 + 2cdem + c^2e^2 - 4cdem \cos^2 \beta$$

where $2\beta = \angle FCD + \angle DEF$. Since $\angle A + \angle C + \angle E = 360°$, $\alpha + \beta = 180°$ and $\cos^2 \alpha = \cos^2 \beta$. Finally,

$$\begin{aligned} d^2g^2h^2 - a^2u^2v^2 &= (a^2d^2m^2 + 2abd^2fm + b^2d^2f^2 - 4abd^2fm \cos^2 \alpha) \\ &\quad - (a^2d^2m^2 + 2a^2cdem + a^2c^2e^2 - 4a^2cdem \cos^2 \beta) \\ &= 2adm(bdf - ace) + (b^2d^2f^2 - a^2c^2e^2) - 4adm(bdf - ace) \cos^2 \alpha = 0 \ , \end{aligned}$$

whence $auv = dgh$ as required.

Solution 3. [Y. Zhao] The proof uses inversion in a circle and directed angles. Recall that, if O is the centre of a circle of radius r, then inversion is that involution $X \leftrightarrow X'$ for which X' is on the ray from O through X and $OX \cdot OX' = r^2$. It is not too hard to check using similar triangles that $\angle OPQ = \angle OQ'P'$ and using the law of cosines that $P'Q' = PQ \cdot (r^2/(OP \cdot OQ))$. For this problem, we make F the centre of the inversion. Then

$$\begin{aligned} 360^\circ &= \angle FAB + \angle BCD + \angle DEF = \angle FAB + \angle BCF + \angle FCD + \angle DEF \\ &= \angle A'B'F + \angle FB'C' + \angle C'D'F + \angle FD'E' = \angle A'B'C' + \angle C'D'E' \end{aligned}$$

whence $\angle C'B'A' = \angle C'D'E'$.

In the following, we suppress the factor r^2. We obtain that

$$\begin{aligned} \frac{A'B'}{B'C'} \cdot \frac{C'D'}{D'E'} &= \left(\frac{AB}{FA \cdot FB} \cdot \frac{FB \cdot FC}{BC}\right) \cdot \left(\frac{CD}{FC \cdot FD} \cdot \frac{FD \cdot FE}{DE}\right) \\ &= \frac{AB}{FA} \cdot \frac{CD}{BC} \cdot \frac{EF}{DE} = 1 \end{aligned}$$

so that $A'B' : B'C' = D'E' : C'D'$. This, along with $\angle C'B'A' = \angle C'D'E'$ implies that $\Delta C'B'A' \sim \Delta C'D'E'$, so that $A'B' : A'C' = D'E' : E'C'$ or $A'B' \cdot E'C' = A'C' \cdot E'D'$.

Therefore

$$\begin{aligned} \frac{AB}{BF} \cdot \frac{FD}{DE} \cdot \frac{EC}{CA} &= \left(\frac{A'B'}{FA' \cdot FB'} \cdot B'F\right) \cdot \left(\frac{1}{F'D'} \cdot \frac{FD' \cdot FE'}{D'E'}\right) \cdot \left(\frac{E'C'}{FE' \cdot FC'} \cdot \frac{FC' \cdot FA'}{C'A'}\right) \\ &= \frac{A'B'}{A'C'} \cdot \frac{E'C'}{E'D'} = 1 \ , \end{aligned}$$

as desired.

Solution 4. [M. Abdeh-Kolahchi] Let A, B, C, D, E, F be points in the complex plane with

$$B - A = a = |a|(\cos\alpha + i\sin\alpha)$$

$$C - B = b = |b|(\cos\beta + i\sin\beta)$$

$$D - C = c = |c|(\cos\gamma + i\sin\gamma)$$

$$E - D = d = |d|(\cos\delta + i\sin\delta)$$

$$F - E = e = |e|(\cos\epsilon + i\sin\epsilon)$$

$$A - F = f = |f|(\cos\phi + i\sin\phi) \ .$$

Modulo 360°, we have that

$$\angle A = \angle FAB \equiv 180^\circ - (\phi - \alpha)$$

$$\angle C = \angle BCD \equiv 180^\circ - (\delta - \beta)$$

$$\angle E = \angle DEF \equiv 180^\circ - (\epsilon - \gamma) \ .$$

Also $a + b + c + d + e + f = 0$ and

$$\begin{aligned} \frac{ace}{bdf} &= \frac{|a||c||e|(\cos\alpha + i\sin\alpha)(\cos\gamma + i\sin\gamma)(\cos\epsilon + i\sin\epsilon)}{|b||d||f|(\cos\beta + i\sin\beta)(\cos\delta + i\sin\delta)(\cos\phi + i\sin\phi)} \\ &= 1(\cos(\alpha - \phi + \delta - \beta + \epsilon - \gamma)) \\ &= \cos(\angle A - 180^\circ + \angle C - 180^\circ + \angle E - 180^\circ) = \cos(-180^\circ) = -1 \ , \end{aligned}$$

whence $ace + bdf = 0$. Therefore,

$$0 = ad(a + b + c + d + e + f) + (ace + bdf) = a(d + e)(c + d) + d(a + f)(a + b) ,$$

whence

$$\frac{a(d+e)(c+d)}{d(a+f)(a+b)} = -1 \Longrightarrow \frac{|a|}{|a+f|} \cdot \frac{|d+e|}{|d|} \cdot \frac{|c+d|}{|a+b|} = 1$$

$$\Longrightarrow \frac{AB}{BF} \cdot \frac{FD}{DE} \cdot \frac{EC}{CA} = 1 .$$

285. (a) Solve the following system of equations:

$$(1 + 4^{2x-y})(5^{1-2x+y}) = 1 + 2^{2x-y+1} ;$$

$$y^2 + 4x = \log_2(y^2 + 2x + 1) .$$

(b) Solve for real values of x:

$$3^x \cdot 8^{x/(x+2)} = 6 .$$

Express your answers in a simple form.

Solution. Let $u = 2x - y$. Then

$$(1 + 4^u)(5^{1-u}) = 1 + 2^{u+1}$$

so that

$$5^{u-1} = \frac{1 + 2^{2u}}{1 + 2^{u+1}} = 2^{u-1} + \frac{1 - 2^{u-1}}{1 + 2^{u+1}} .$$

Thus,

$$5^{u-1} - 2^{u-1} = \frac{1 - 2^{u-1}}{1 + 2^{u+1}} .$$

When $u > 1$, the left side of this equation is positive while the right is negative; when $u < 1$, the reverse is true. Hence, the only possible solution is $u = 1$, which checks out.

Substituting for x leads to

$$y^2 + 2y + 2 = \log_2(y^2 + y + 2) .$$

Since $y^2 + y + 2 = (y + \frac{1}{2})^2 + \frac{7}{4} > 0$, the right side is defined and is in fact positive. Let

$$\phi(y) = y^2 + 2y + 2 - \log_2(y^2 + y + 2) .$$

Then

$$\phi'(y) = \frac{2y(y+1)^2 + 4(y+1) - (\log_2 e)(2y+1)}{y^2 + y + 2} .$$

$$\phi'(y) = 0 \Longleftrightarrow (y+1)^2 = -\left((2 - \log_2 e) + \frac{4 - \log_2 e}{2y}\right) .$$

From the graphs of the two sides of the equation, we see that the left side and the right side have opposite signs when $y > 0$ and become equal for exactly one value of y. It follows that $\phi'(y)$ changes sign exactly once so that $\phi(y)$ decreases and then increases. Thus, $\phi(y)$ vanishes at most twice. Indeed, $\phi(-2) = \phi(-1) = 0$, and so $(x, y) = (0, -1), (-\frac{1}{2}, -2)$ are the only solutions of the equation.

(b) The equation can be rewritten

$$1 = 3^{1-x} 2^{2(1-x)/(x+2)}$$

whence

$$0 = (1 - x)(\log 3 + (2/(x + 2)) \log 2) \ .$$

Thus, either $x = 1$ or $0 = \log_2 3 + 2/(x + 2)$. The latter leads to

$$x = -2(1 + \log_3 2) = -2(\log_3 6) = -\log_3 36 \ .$$

286. Construct inside a triangle ABC a point P such that, if X, Y, Z are the respective feet of the perpendiculars from P to BC, CA, AB, then P is the centroid (intersection of the medians) of triangle XYZ.

Solution 1. Let AU, BV, CW be the medians of triangle ABC and let AL, BM, CN be their respective images in the bisectors of angles A, B, C. Since AU, BV, CW intersect in a common point (the centroid of ΔABC). AL, BM, CN must intersect in a common point P. This follows from the sine version of Ceva's theorem and its converse. Let X, Y, Z be the respective feet of the perpendiculars from P to sides BC, AC, AB.

Let I, J, K be the respective feet of the perpendiculars from the centroid G to the sides BC, AC and AB. The quadrilateral $PYAZ$ is the image of the quadrilateral $GJAK$ under a reflection in the angle bisector of A followed by a dilation with centre A and factor AP/AG. Hence $PY : PZ = GK : GI$. Since triangles AGB and AGC have the same area,

$$AB \cdot GK = AC \cdot GJ \Longrightarrow PY : PZ = AC : AB = b : c \ .$$

Applying a similar argument involving PX, we find that

$$PX : PY : PZ = a : b : c \ .$$

Let $PX = ae$, $PY = be$, $PZ = ce$. Then, since $\angle XPY + \angle ACB = 180°$,

$$[PXY] = \frac{1}{2}abe^2 \sin \angle XPY = e^2 \left(\frac{1}{2}ab \sin C \right) = e^2[ABC] \ .$$

Similarly, $[PYZ] = [PZX] = e^2[ABC] = [PXY]$, whence P must be the centroid of triangle XYZ.

Solution 2. [M. Lipnowski] Erect squares $ARSB$, $BTUC$, $CVWA$ externally on the edges of the triangle. Suppose that RS and VW intersect at A', RS and TU at B' and TU and UW at C'.

We establish that AA', BB' and CC' are concurrent. They are cevians in the triangle $A'B'C'$. We have that

$$\begin{aligned} \frac{\sin \angle RA'A}{\sin \angle WA'A} \cdot \frac{\sin \angle VC'C}{\sin \angle UC'C} \cdot \frac{\sin \angle TB'B}{\sin \angle SB'B} \\ &= \frac{(AR/AA')}{(AW/AA')} \cdot \frac{(VC/CC')}{(UC/CC')} \cdot \frac{(TB/BB')}{(BS/BB')} \\ &= \frac{AR}{AW} \cdot \frac{VC}{UC} \cdot \frac{TB}{BC} = \frac{c}{b} \cdot \frac{b}{a} \cdot \frac{a}{c} = 1 \ . \end{aligned}$$

Hence AA', BB', CC' intersect in a point P by the converse to Ceva's Theorem. P is the desired point.

To prove that this works, we first show that $PX : PY : PZ = a : b : c$, and then that $[XPY] = [YPZ] = [ZPX]$. Observe that, since $\Delta PZA \sim \Delta ARA'$ and $\Delta PYA \sim \Delta AWA'$,

$$\frac{PY}{PZ} = \frac{PY(AA'/PA)}{PZ(AA'/PA)} = \frac{AW}{AR} = \frac{b}{c} \ ,$$

and similarly that $PX : PZ = a : c$. Now

$$\angle XPY = 360° - \angle PXC - \angle PYC - \angle XCY = 180° - \angle XCY = 180° - \angle ACB \ ,$$

so that $[XPY] = \frac{1}{2}PX \cdot PY \sin \angle XPY = \frac{1}{2}PX \cdot PY \sin \angle ACB$. We find that

$$[XPY] : [YPZ] : [ZPX] = \frac{1}{2}PX \cdot PY \sin \angle ACB : \frac{1}{2}PY \cdot PZ \sin \angle ACB : \frac{1}{2}PZ \cdot PX \sin \angle ABC$$
$$= \frac{1}{2}ab \sin C : \frac{1}{2}bc \sin A : \frac{1}{2}ca \sin B = [ABC] : [ABC] : [ABC] = 1 : 1 : 1 \ .$$

Hence $[XPY] = [YPZ] = [ZPX] = \frac{1}{3}[XYZ]$, so that the altitudes of these triangle from P to the sides of triangle XYZ are each one-third of the corresponding altitudes for triangle XYZ. Hence P must be the centroid of triangle XYZ.

Comment. A. Zhang and Y. Zhao gave the same construction. Zhang first gave an argument that P, being the centroid of triangle XYZ is characterized by $PX : PY : PZ = a : b : c$. This is a result of the characterization $[XPY] = [YPZ] = [ZPX]$ and the law of sines, with the argument similar to Lipnowski's. Zhao used the fact that $PX : PY : PZ = BC : CA : AB$ and that the vectors $\overrightarrow{PX}$, $\overrightarrow{PY}$, $\overrightarrow{PZ}$ were dilated versions of $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ after a $90°$ rotation, so that $\overrightarrow{PX} + \overrightarrow{PY} + \overrightarrow{PZ} = \overrightarrow{O}$.

287. Let M and N be the respective midpoints of the sides BC and AC of the triangle ABC. Prove that the centroid of the triangle ABC lies on the circumscribed circle of the triangle CMN if and only if

$$4 \cdot |AM| \cdot |BN| = 3 \cdot |AC| \cdot |BC| \ .$$

Solution 1.

$$4|AM||BN| = 3|AC||BC| \Longleftrightarrow 12|AM||GN| = 12|AN||MC| \Longleftrightarrow |AM| : |MC| = |AN| : |GN|$$

$$\Longleftrightarrow \Delta AMC \sim \Delta ANG \Longleftrightarrow \angle AMC = \angle ANG$$

$\Longleftrightarrow GMGN$ is concyclic.

Solution 2. [A. Zhang] Since M and N are respective midpoints of BC and AC, $[ABC] = 4[NMC]$, so that

$$[ABMN] = \frac{3}{4}[ABC] = \frac{3}{8}|AC||BC| \sin \angle ACB \ .$$

However, $[ABMN] = \frac{1}{2}|AM||BN| \sin \angle NGM$ (why?). Hence

$$4|AM||BN| \sin \angle NGM = 3|AC||BC| \sin \angle ACB \ .$$

Observe that G lies inside triangle ABC, and so lies within the circumcircle of this triangle. Hence $\angle NGM = \angle AGB > \angle ACB$. We deduce that

$$4|AM||BN| = 3|AC||BC| \Longleftrightarrow \sin \angle NGM = \sin \angle ACB \Longleftrightarrow \angle NGM + \angle ACB = 180°$$

$\Longleftrightarrow CMGN$ is concyclic.

288. Suppose that $a_1 < a_2 < \cdots < a_n$. Prove that

$$a_1 a_2^4 + a_2 a_3^4 + \cdots + a_n a_1^4 \geq a_2 a_1^4 + a_3 a_2^4 + \cdots + a_1 a_n^4 \ .$$

Solution. The result is trivial for $n = 2$. To deal with the $n = 3$ case, observe that, when $x < y < z$,

$$(xy^4 + yz^4 + zx^4) - (yx^4 + zy^4 + xz^4) = (1/2)(z - x)(y - x)(z - y)[(x + y)^2 + (x + z)^2 + (y + z)^2] \geq 0 \ .$$

As an induction hypothesis, assume that the result holds for the index $n \geq 3$. Then

$$\begin{aligned}(a_1a_2^4 + a_2a_3^4 + \cdots + a_na_{n+1}^4 + a_{n+1}a_1^4) - (a_2a_1^4 + a_3a_2^4 + \cdots + a_{n+1}a_n^4 + a_1a_{n+1}^4) \\ = (a_1a_2^4 + a_2a_3^4 + \cdots + a_na_1^4) - (a_2a_1^4 + a_3a_2^4 + \cdots + a_1a_n^4) \\ + (a_1a_n^4 + a_na_{n+1}^4 + a_{n+1}a_1^4) - (a_na_1^4 + a_{n+1}a_n^4 + a_1a_{n+1}^4) \geq 0 \ ,\end{aligned}$$

as desired.

289. Let $n(r)$ be the number of points with integer coordinates on the circumference of a circle of radius $r > 1$ in the cartesian plane. Prove that

$$n(r) < 6\sqrt[3]{\pi r^2} \ .$$

Solution. Let $A = \pi r^2$ be the area of the circle, so that the right side of the inequality is $6A^{1/3}$. We observe that $A > 3$, $\pi^2 < (22/7)^2 < 10 < (2.2)^3$.

$$\begin{aligned}6A^{1/3} - 2\pi^{2/3}A^{1/3} &= (6 - 2\pi^{2/3})A^{1/3} > (6 - 2 \times 10^{1/3})A^{1/3} \\ &> (6 - 4.4) \times 3^{1/3} > 1.6 \times 1.25 = 2 \ ,\end{aligned}$$

so that there is an even integer k for which

$$6 = 2 \times 3^{2/3} \times 3^{1/3} < 2\pi^{2/3}A^{1/3} < k < 6A^{1/3} \ .$$

In particular, $8\pi^2 A < k^3$.

Let $P_1P_2\cdots P_k$ be a regular $k-$gon inscribed in the circle. Locate the vertices so that none have integer coordinates. (How?) Identify $P_{k+1} = P_1$ and $P_{k+2} = P_2$, and let $\mathbf{v_i} = \overrightarrow{P_iP_{i+1}}$ for $1 \leq i \leq k$. Observe that $\mathbf{v_i}$ has length less than $2\pi r/k = (2/k)(\pi A)^{1/2}$. Then, for each i, the area of triangle $P_iP_{i+1}P_{i+2}$ is equal to

$$\frac{1}{2}|\mathbf{v_i} \times \mathbf{v_{i+1}}| = \frac{1}{2}|\mathbf{v_i}||\mathbf{v_{i+1}}|\sin(2\pi/k) < \frac{1}{2} \times \frac{4}{k^2} \times \pi A \times \frac{2\pi}{k} = \frac{1}{2} \times \frac{8\pi^2}{k^3} \times A < \frac{1}{2} \ .$$

Suppose, if possible, that the arc joining P_i and P_{i+2} (through P_{i+1}) contains points U, V, W, each with integer coordinates. Then, if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are the corresponding vectors for these points, then $|(\mathbf{v}-\mathbf{u}) \times (\mathbf{w}-\mathbf{u})|$ must be a positive integer, and so the area of triangle UVW must be at least $1/2$. But each of the sides of triangle UVW has length less than the length of P_iP_{i+2} and the shortest altitude of triangle UVW is less than the altitude of triangle $P_iP_{i+1}P_{i+2}$ from P_{i+1} to side P_iP_{i+2}. Thus,

$$\frac{1}{2} \leq [UVW] \leq [P_iP_{i+1}P_{i+2}] < \frac{1}{2} \ ,$$

a contradiction. Hence, each arc P_iP_{i+2} has at most two points with integer coordinates. The whole circumference of the circle is the union of $k/2$ nonoverlapping such arcs, so that there must be at most k points with integer coordinates. The result follows.