Problems for January, 2005

Please mail your solutions to Ms. Valeria Pandelieva, 641 Kirkwood Avenue, Ottawa, ON K1Z 5X5 no later than **February 15, 2005**. Please make sure that your name, full mailing address and email address appears on the front page of your solutions. If you do not write your family name last, please indicate it with an asterisk (*).

353. The two shortest sides of a right-angled triangle, a and b, satisfy the inequality:

$$\sqrt{a^2 - 6a\sqrt{2} + 19} + \sqrt{b^2 - 4b\sqrt{3} + 16} \leq 3 \ .$$

Find the perimeter of this triangle.

354. Let ABC be an isosceles triangle with $AC = BC$ for which $|AB| = 4\sqrt{2}$ and the length of the median to one of the other two sides is 5. Calculate the area of this triangle.

355. (a) Find all natural numbers k for which $3^k - 1$ is a multiple of 13.

(b) Prove that for any natural number k, $3^k + 1$ is not a multiple of 13.

356. Let a and b be real parameters. One of the roots of the equation $x^{12} - abx + a^2 = 0$ is greater than 2. Prove that $|b| > 64$.

357. Consider the circumference of a circle as a set of points. Let each of these points be coloured red or blue. Prove that, regardless of the choice of colouring, it is always possible to inscribe in this circle an isosceles triangle whose three vertices are of the same colour.

358. Find all integers x which satisfy the equation

$$\cos\left(\frac{\pi}{8}(3x - \sqrt{9x^2 + 160x + 800})\right) = 1 \ .$$

359. Let ABC be an acute triangle with angle bisectors AA_1 and BB_1, with A_1 and B_1 on BC and AC, respectively. Let J be the intersection of AA_1 and BB_1 (the incentre), H be the orthocentre and O the circumcentre of the triangle ABC. The line OH intersects AC at P and BC at Q. Given that C, A_1, J and B_1 are vertices of a concyclic quadrilateral, prove that $PQ = AP + BQ$.