Problems for September, 2005

Please send your solution to
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no later than October 31, 2005 It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

Notes: The absolute value $|x|$ is equal to x when x is nonnegative and $-x$ when x is negative; always $|x| \geq 0$. The *floor of* x, denoted by $\lfloor x \rfloor$ is equal to the greatest integer that does not exceed x. For example, $\lfloor 5.34 \rfloor = 5$, $\lfloor -2.3 \rfloor = -3$ and $\lfloor 5 \rfloor = 5$. A geometric figure is said to be *convex* if the segment joining any two points inside the figure also lies inside the figure.

402. A point M is located on the side AB of triangle ABC for which $BM = 2AM$. Given that $\angle ACM = 15^\circ$ and $\angle BMC = 60^\circ$, calculate the angles of the triangle.

403. Let $f(x) = |1 - 2x| - 3|x + 1|$ for real values of x.

    (a) Determine all values of the real parameter a for which the equation $f(x) = a$ has two different roots u and v that satisfy $2 \leq |u - v| \leq 10$.

    (b) Solve the equation $f(x) = \lfloor x/2 \rfloor$.

404. Several points in the plane are said to be *in general position* if no three are collinear.

    (a) Prove that, given 5 points in general position, there are always four of them that are vertices of a convex quadrilateral.

    (b) Prove that, given 400 points in general position, there are at least 80 nonintersecting convex quadrilaterals, whose vertices are chosen from the given points. (Two quadrilaterals are nonintersecting if they do not have a common point, either in the interior or on the perimeter.)

    (c) Prove that, given 20 points in general position, there are at least 969 convex quadrilaterals whose vertices are chosen from these points. (**Bonus:** Derive a formula for the number of these quadrilaterals given n points in general position.)

405. Suppose that a permutation of the numbers from 1 to 100, inclusive, is given. Consider the sums of all triples of consecutive numbers in the permutation. At most how many of these sums can be odd?

406. Let a, b. c be natural numbers such that the expression

$$\frac{a+1}{b} + \frac{b+1}{c} + \frac{c+1}{a}$$

    is also equal to a natural number. Prove that the greatest common divisor of a, b and c, $\gcd(a, b, c)$, does not exceed $\sqrt[3]{ab + bc + ca}$, *i.e.*,

$$\gcd(a, b, c) \leq \sqrt[3]{ab + bc + ca}\ .$$

407. Is there a pair of natural numbers, x and y, for which

    (a) $x^3 + y^4 = 2^{2003}$?

    (b) $x^3 + y^4 = 2^{2005}$?

    Provide reasoning for your answers to (a) and (b).

408. Prove that a number of the form $a000 \cdots 0009$ (with $n + 2$ digits for which the first digit a is followed by n zeros and the units digit is 9) cannot be the square of another integer.