Solutions to March problems.

367. Let a and c be fixed real numbers satisfying $a \leq 1 \leq c$. Determine the largest value of b that is consistent with the condition

$$a + bc \leq b + ac \leq c + ab \ .$$

Solution. Since $(b + ac) - (a + bc) = (a - b)(c - 1)$ and $(c + ab) - (b + ac) = (c - b)(1 - a)$, the given inequalities are equivalent to $(a - b)(c - 1) \geq 0$ and $(c - b)(1 - a) \geq 0$.

If $a = c = 1$, then the inequalites hold for any value of b, and there is no maximum value. If $a < 1 = c$, then the first inequality is automatic and the inequalities hold if and only if $b \leq c = 1$. If $1 < c$, then $a \leq 1 < c$ and the two inequalities are equivalent to $a \geq b$ and $c \geq b$, which both hold if and only if $a \geq b$. Thus, when $1 < c$, the maximum value of b for which the inequalities hold is a. ♠

368. Let A, B, C be three distinct points of the plane for which $AB = AC$. Describe the locus of the point P for which $\angle APB = \angle APC$.

Solution 1. We observe that P cannot be any of A, B, C, as the angles become degenerate, so that A, B and C must be excluded from the locus. Suppose first that A, B, C are collinear, so that A is the midpoint of BC. For a point P on the locus for which the triangle PBC is nondegenerate, the median PA of the triangle PBC bisects the angle BPC. Therefore $PB = PC$ (note that $PB : PC = AB : AC = 1 : 1$) and so P must lie on the right bisector of BC. Conversely, any point on the righ bisector save B is on the locus. If the triangle PBC is degenerate, then P must lie on the line BC outside of the closed interval BC (in which case $\angle APB = \angle APC$).

Henceforth, assume that the points ABC are not collinear. The locus does not contain any of the points A, B and C. If P is a point on the open arc BC of the circumcircle of ABC that does not contain A, then $ABPC$ is concyclic and

$$\angle APB = \angle ACB = \angle ABC = \angle APC \ .$$

(More briefly, the angles subtended at P be the equal chords AB and AC are equal. Why are they not supplementary?) If P is any point, except A, on the right bisector of BC, then, by a reflection in this bisector, we see that $\angle APB = \angle APC$. Finally, if P is any point on the BC outside of the closed segment BC, then $\angle APB = \angle APC$ since P, B, C are collinear. Thus, the locus must contain the following three sets: (1) all points on the open arc BC of the circumcircle of ABC that does not contain A; (2) all points on the right bisector of BC except for A; (3) all points on the line BC than do not lie between B and C inclusive.

We show that there are no further points in the locus. Suppose that P lies in the angle formed by AB and AC that includes the right bisector of BC, but that P does not lie on this bisector. Let Q (distinct from P) be the reflection of P in the right bisector. Then $\angle APB = \angle AQC$ and $\angle APC = \angle AQB$. Suppose that $\angle APB = \angle APC$. Then $\angle APC = \angle AQC$ and $\angle APB = \angle AQB$, so that A, B, P, Q, C are concyclic and we must have situation (i). If P lies in the angle exterior to triangle ABC determined by AB and AC produced, then it can be checked that one of the angles APB and APC properly contains the other. The result follows.

Solution 2. If A, B, C are collinear, wolog suppose that P is a point of the locus not on BC for which $PB < PC$ and D is the reflected image of B with respect to PA. Then $AD = BA = AC$ and D lies on the segment PC. Hence $\angle PBA + \angle PCB = \angle PDA + \angle DCA = \angle PDA + \angle ADC = 180°$, so that $\angle BPC = 0°$, a contradiction. Hence P must lie on BC or the right bisector of BC, We can eliminate from the locus all points on the closed segment BC as well as the point A.

Suppose A, B and C are not collinear. Let P be a point on the locus. Consider triangle ABP and ACP. Since $AB = AC$, AP is common and the (noncontained) corresponding angles APB and APC are equal, we have the ambiguous (SSA) case and so either triangles APB and APC are congruent, or else $\angle ABP + \angle ACP = 180°$. If the triangles are conguent, then $PB = PC$ and P lies on the right bisector of BC, If $\angle ABP + \angle ACP = 180°$, then there are two possibilities. Either B and C lie on the same side of AP, in which case P, B, C are collinear or B and C lie on opposite sides of AP, in which case $ABPC$ is concyclic.

Hence the locus is contained in the union of the right bisector of BC, that part of the line BC not between B and C and the arc BC of the circumcircle of triangle ABC not containing A. Conversely, it is straightforward to verify that every point in this union, except for A, B and C is on the locus.

Comment. Another way in is to apply the law of sines on triangles ABP and ACP and note that

$$\frac{\sin ABP}{|AP|} = \frac{\sin \angle APB}{|AB|} = \frac{\sin \angle APC}{|AC|} = \frac{\sin \angle ACP}{|AP|} \ ,$$

so that $\sin \angle ABP = \sin \angle ACP$. Thus, either $\angle ABP = \angle ACP$ or $\angle ABP + \angle ACP = 180°$.

369. $ABCD$ is a rectangle and APQ is an inscribed equilateral triangle for which P lies on BC and Q lies on CD.

(a) For which rectangles is the configuration possible?

(b) Prove that, when the configuration is possible, then the area of triangle CPQ is equal to the sum of the areas of the triangles ABP and ADQ.

Solution 1. (a) Let the configuration be given and let the length of the side of the equilateral triangle be 1. Suppose that $\angle BAP = \theta$. Then $|AB| = \cos\theta$ and $|AD| = \cos(30° - \theta)$. Observe that $0 \le \theta \le 30°$. Then

$$\frac{|AD|}{|AB|} = \frac{\cos 30° \cos\theta + \sin 30° \sin\theta}{\cos\theta} = \frac{\sqrt{3} + \tan\theta}{2} \ .$$

Since $0 \le \tan\theta \le 1/\sqrt{3}$, it follows that

$$\frac{\sqrt{3}}{2} \le \frac{|AD|}{|AB|} \le \frac{2}{\sqrt{3}} \ .$$

(Alternatively, note that $\cos(30° - \theta)$ increases and $\cos\theta$ decreases with θ, so that

$$\frac{\cos 30°}{\cos 0°} \le \frac{|AD|}{|AB|} \le \frac{\cos 0°}{\cos 30°} \ .)$$

Conversely, supposing that this condition is satisfied, we can solve $|AD|/|AB| = \frac{1}{2}(\sqrt{3} + \tan\theta)$ for a value of $\theta \in [0, 30°]$ and determine a configuration for which $\angle BAP = \theta$ and $\angle DAQ = 30° - \theta$. It can be checked that $AP = AQ$ (do this!).

(b) Suppose that the configuration is given, and that L, M and N are the respective midpoints of AP, PQ and QA. Wolog, let the lengths of these three segments be 2. Then BL, CM and DN all have length 1 (why?). Since $\angle BLP = 2\theta$, $\angle CMQ = 60° + 2\theta$ and $\angle DNQ = 60° - 2\theta$, we find that (from the areas of triangles like BLP),

$$\begin{aligned}[CPQ] - [ADQ] - [ABP] &= \sin(60° + 2\theta) - \sin(60° - 2\theta) - \sin 2\theta \\ &= 2\cos 60° \sin 2\theta - \sin 2\theta = 0 \ ,\end{aligned}$$

so that the result holds.

Comment. One can also get the areas of the corner right triangles by taking half the product of their arms. For example,

$$[BAP] = \frac{1}{2}(2\sin\theta)(2\cos\theta) = 2\sin\theta\cos\theta = \sin 2\theta \ .$$

370. A deck of cards has nk cards, n cards of each of the colours C_1, C_2, $\cdots$, C_k. The deck is thoroughly shuffled and dealt into k piles of n cards each, P_1, P_2, $\cdots$, P_k. A game of solitaire proceeds as follows:

The top card is drawn from pile P_1. If it has colour C_i, it is discarded and the top card is drawn from pile P_i. If it has colour C_j, it is discarded and the top card is drawn from pile P_j. The game continues in this way, and will terminate when the nth card of colour C_1 is drawn and discarded, as at this point, there are no further cards left in pile P_1. What is the probability that every card is discarded when the game terminates?

Solution. We begin by determining a one-one correspondence between plays of the game and the $(nk)!$ arrangements of the nk cards. For each play of the game, we set the cards aside in the order than they appear. If the game is finishes with the last card, we go through the whole deck and obtain an arrangement in which the last card has colour C_1. If the game finishes early, then we have exhausted the pile P_1, but not all of the remaining pile; all the colours C_1's will have appeared among the first $nk - 1$ cards. We continue the arrangement by dealing out in order all the cards in the pile P_2, then all the cards in the pile P_3 and so on.

Conversely, suppose that we have an arrangement of the nk cards. We reconstruct a game. Look at all the cards up to the last card of colour C_1. Suppose that it contains x_i cards of colour C_i; then that means that there are $n - x_i$ cards that should not be turned over in pile P_i. Take the last $n - x_k$ cards of the arrangement and place them in pile P_k, and then the next last $n - x_{k-1}$ cards and place them in pile P_{k-1}, and so on until we come down to pile P_2. We have backed up in the arrangement to the last card of colour C_1, and its predecessor determines from which pile it was drawn; restore it to that pile. For $2 \leq i \leq x_1 + x_2 + \cdots + x_k$, place the ith card in the arrangement on the pile of the colour of the $(i-1)$th card. Finally, when we get to the first card of the arrangement, all the piles except P_1 have been restored to n cards; place this card on P_1.

Thus, each game determines an arrangement, and each arrangement a game. Therefore, the desired probability is the probability that in an arbitrary arrangement, the last card has colour C_1. As the probability is the same for each of the colours, the desired probability is $1/k$.

371. Let X be a point on the side BC of triangle ABC and Y the point where the line AX meets the circumcircle of triangle ABC. Prove or disprove: if the length of XY is maximum, then AX lies between the median from A and the bisector of angle BAC.

Solution. [F. Barekat] Wolog, suppose that $AB \geq AC$. Let M be the midpoint of BC and N where AM intersect the circumcircle. Let P on BC be the foot of the angle bisector of $\angle BAC$ and let AP intersect the circumcircle at Q. Then BC is partitioned into three segments: BM, MP, PC.

Suppose that X is on the segment BM and that AX meets the circumcircle in Y. Let U on the segment MC satisfy $XM = MU$ and let AU meet the circumcircle in V then

$$AX \cdot XY = BX \cdot XC = CU \cdot UB = AU \cdot UV \ .$$

Now $AX \geq AU$ (one way to see this is to drop a perpendicular from A to XU and use Pythagoras' theorem). Hence $XY \leq UV$, so the maximizing point must lie between M and C.

Now let X lie between P and C. Construct R so that $PXYR$ is a parallelogram. Since Q is the midpoint of the arc BC, the tangent at Q to the circumcircle is parallel to BC and so R lies within the circle, $\angle PQR \leq \angle AQY$ and $\angle QPR = \angle QAY$. Therefore

$$\begin{aligned} \angle PRQ &= 180^\circ - \angle QPR - \angle PQR \geq 180^\circ - \angle QAY - \angle AQY \\ &= \angle AYQ = \angle ACQ = \angle ACB + \angle BCQ \\ &= \angle ACB + \angle BAQ = \angle ACB + \frac{1}{2}\angle BAC \\ &= \frac{1}{2}(2\angle ACB + \angle BAC) \geq \frac{1}{2}(\angle ACB + \angle ABC + \angle BAC) = 90^\circ \ . \end{aligned}$$

Therefore $PQ \geq PR = XY$. It follows that XY assumes its maximum value when X is between M and P, as desired.

372. Let b_n be the number of integers whose digits are all $1, 3, 4$ and whose digits sum to n. Prove that b_n is a perfect square when n is even.

Solution 1. It is readily checked that $b_1 = b_2 = 1$, $b_3 = 2$ and $b_4 = 4$. Consider numbers whose digits sum to $n \geq 5$. There are b_{n-1} of them ending in 1, b_{n-3} of them ending in 3, and b_{n-4} of them ending in 4. We prove by induction, that for each positive integer m,

$$b_{2m} = f_{m+1}^2 \qquad \text{and} \qquad b_{2m-1} = f_{m+1}f_m \ ,$$

where $\{f_n\}$ is the Fibonacci sequences defined by $f_0 = 0, f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$.

The result holds for $m = 1$. Suppose that it holds up to $m = k$. Then

$$\begin{aligned} b_{2k+1} &= b_{2k} + b_{2k-2} + b_{2k-3} = f_{k+1}^2 + f_k^2 + f_k f_{k-1} \\ &= f_{k+1}^2 + f_k(f_k + f_{k-1}) = f_{k+1}^2 + f_k f_{k+1} \\ &= f_{k+1}(f_{k+1} + f_k) = f_{k+1}f_{k+2} \ , \end{aligned}$$

and

$$\begin{aligned} b_{2(k+1)} &= b_{2k+1} + b_{2k-1} + b_{2k-2} = f_{k+2}f_{k+1} + f_{k+1}f_k + f_k^2 \\ &= f_{k+2}f_{k+1} + (f_{k+1} + f_k)f_k \\ &= f_{k+2}f_{k+1} + f_{k+2}f_k = f_{k+2}(f_{k+1} + f_k) = f_{k+2}^2 \ . \end{aligned}$$

The result follows.

Solution 2. As before, $b_n = b_{n-1} + b_{n-3} + b_{n-4}$. From this, we see that

$$b_n = (b_{n-2} + b_{n-4} + b_{n-5}) + (b_{n-2} - b_{n-5} - b_{n-6}) + b_{n-4} = 2b_{n-2} + 2b_{n-4} - b_{n-6} \ ,$$

for $n \geq 7$. Also, for $n \geq 4$,

$$f_n^2 = (f_{n-1} + f_{n-2})^2 = 2f_{n-1}^2 + 2f_{n-2}^2 - (f_{n-1} - f_{n-2})^2 = 2f_{n-1}^2 + 2f_{n-2}^2 - f_{n-3}^2 \ .$$

By induction, it can be shown that $b_{2m} = f_{m+1}^2$ for each positive integer m (do it!).

373. For each positive integer n, define

$$a_n = 1 + 2^2 + 3^3 + \cdots + n^n \ .$$

Prove that there are infinitely many values of n for which a_n is an odd composite number.

Solution. Modulo 3, it can be verified that $n^n \equiv 0$ for $n \equiv 0 \pmod 3$, $n^n \equiv 1$ for $n \equiv 1, 2, 4 \pmod 6$, and $n^n \equiv 2$ for $n \equiv 5 \pmod 6$. It follows from this that the sum of any six consecutive values of n^n is congruent to $2 \pmod 3$, and so the sum of any eighteen consecutive values of n^n is congruent to $0 \pmod 3$. Since such a sum contains nine odd summands, it must be odd. The sum of any thirty-six consecutive values of n^n contains eighteen odd summands and so is even. It follows that the sum of any thirty-six consecutive values of n^n is a multiple of 6.

It is readily checked that $a_n \equiv 0 \pmod 3$ when $n = 4, 7, 14, 15, 17, 18$. Observe that a_4, a_7, a_{15} are even and a_{14}, a_{17}, a_{18} are odd. Hence a_n is an odd multiple of 3 whenever $n \equiv 14, 17, 18, 22, 25, 33 \pmod{36}$. These numbers are all odd and composite.

Comment. A similar argument can be had for any odd prime p. What is the period of n^n?