Solutions for October

458. Let ABC be a triangle. Let A_1 be the reflected image of A with axis BC, B_1 the reflected image of B with axis CA and C_1 the reflected image of C with axis AB. Determine the possible sets of angles of triangle ABC for which $A_1B_1C_1$ is equilateral.

Solution. We establish a preliminary result: *For any angle θ,*

$$\cos 3\theta = \cos\theta - 4\cos\theta\sin^2\theta \ . \tag{1}$$

This is true, since

$$\begin{aligned}\cos 3\theta &= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \\ &= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta \\ &= \cos^3\theta - 3\sin^2\theta\cos\theta = \cos\theta(\cos^2\theta - 3\sin^2\theta) \\ &= \cos\theta(1 - 4\sin^2\theta) \ .\end{aligned}$$

Let a, b, c be the sides of the triangle and α, β, γ be the respective angles opposite these sides. Since the triangles A_1BC, AB_1C and ABC_1 are all congruent to the triangle ABC, we have that $\angle C_1AB_1 = 3\alpha$ or $|2\pi - 3\alpha|$, $\angle A_1BC_1 = 3\beta$ or $|2\pi - 3\beta|$, and $\angle B_1CA_1 = 3\gamma$ or $|2\pi - 3\gamma|$.

Applying the Cosine Law to triangle B_1CA_1 yields that

$$A_1B_1^2 = a^2 + b^2 - 2ab\cos 3\gamma \qquad (2) \ .$$

Where R is the circumradius of triangle ABC, the Cosine and the Sine Laws applied to that triangle yields that

$$2ab\cos\gamma = a^2 + b^2 - c^2 \tag{3}$$

and

$$\sin\gamma = \frac{c}{2R} \qquad (4) \ .$$

Applying (1), (3), (4) to (1) yields that

$$\begin{aligned}A_1B_1^2 &= a^2 + b^2 - 2ab\cos 3\gamma = a^2 + b^2 - 2ab\cos\gamma(1 - 4\sin^2\gamma \\ &= a^2 + b^2 - (a^2 + b^2 - c^2)(1 - 4\sin^2\gamma) \\ &= (a^2 + b^2)(1 - 1 + 4\sin^2\gamma) + c^2(1 - 4\sin^2\gamma) \\ &= (a^2 + b^2 - c^2)(4\sin^2\gamma) + c^2 = \frac{(a^2 + b^2 - c^2)c^2}{R^2} + c^2 = \frac{c^2}{R^2}(a^2 + b^2 - c^2 + R^2) \ .\end{aligned}$$

Similarly,

$$B_1C_1^2 = \frac{a^2}{R^2}(b^2 + c^2 - a^2 + R^2)$$

and

$$C_1A_1^2 = \frac{b^2}{R^2}(c^2 + a^2 - b^2 + R^2) \ .$$

It follows that $A_1B_1 = B_1C_1$ if and only if

$$c^2(a^2 + b^2 - c^2 + R^2) - a^2(b^2 + c^2 - a^2 + R^2) = 0 \ .$$

Factoring the left side yields that

$$(c^2 - a^2)(R^2 + b^2 - a^2 - c^2) = 0 \ .$$

The equality of other pairs of sides can be similarly handled. Thus, triangle $A_1B_1C_1$ is equilateral if and only if the following system of three equations is valid:

$$(c^2 - a^2)(R^2 + b^2 - a^2 - c^2) = 0 \ ;$$

$$(a^2 - b^2)(R^2 + c^2 - b^2 - a^2) = 0 \ ;$$

$$(b^2 - c^2)(R^2 + a^2 - b^2 - c^2 = 0 \ .$$

If a is unequal to both b and c, then

$$R = a^2 + c^2 - b^2 = b^2 + a^2 - c^2$$

so that $b = c$. Hence, the triangle is isosceles in any case.

Wolog, assume that $b = c$. Then from the middle equation, we obtain that $(a^2 - b^2)(R^2 - a^2) = 0$. Therefore, either $a = b$, in which case the triangle is equilateral, or $R = a$ and $\sin\alpha = a/2R = 1/2$. Therefore, $\alpha = 30^\circ$ or $\alpha = 150^\circ$. Thus, there are three possible sets of angles fo the triangle ABC: $(60^\circ, 60^\circ, 60^\circ)$, $(30^\circ, 75^\circ, 75^\circ)$ and $(150^\circ, 15^\circ, 15^\circ)$.

459. At an International Conference, there were exactly 2006 participants. The organizers observed that: (1) among any three participants, there were two who spoke the same language; and (2) every participant spoke at most 5 languages. Prove that there is a group of at least 202 participants who speak the same language.

Solution 1. Consider an arbitrary participant, a. Suppose, first, that a can communicate with all other participants. Then, as a speaks at most five languages, there will be at least $\lceil 2006/5 \rceil = 402 > 202$ who speak one of the five languages.

On the other hand, if there is a participant, b, with whom a cannot communicate, then out of the remaining 2004 people, everybody should be able to communicate with either a or b. Thus, one of the pair, say a, can communicate with at least 1002 people. Since a speaks at most five languages, one of these five must be spoken by at least $lceil 1002/5\rceil = 201$ of these 1002 people. Including a among the speakers of this language yields the result.

Solution 2. [D. Dziabenko] Suppose, if possible, that the result is false. So, no language is spoken by 202 people. Let p be any person in the group. This person p spoke at least five languages, and can share each language with at most 200 people. Let P be the set of people with whom p shares at least one language; the set P has at most 1000 individuals.

Suppose that $q \notin P$. By a similar argument, the set Q of individuals with whom q shares at least one language contains at most 1000 people. Thus the set $R = P \bigcup Q \bigcup \{p\} \bigcup \{q\}$ contains at most 2002 people. Let $r \notin R$. Then, since $r \notin P$, r and p do not share a language. Since $r \notin Q$, r and q do not share a language. Since $q \notin P$, p and q do not share a language. Then $\{p, q, r\}$ is a triplet that violates condition (1), and we obtain a contradiction. Hence, there must be 202 people with a common language.

460. Given two natural numbers x and y for which

$$3x^2 + x = 4y^2 + y \ ,$$

prove that their positive difference is a perfect square. Determine a nontrivial solution of this equation.

Solution 1. [D. Dziabenko] Since $3(x-y)(x+y+1) = y^2$, $x > y$. Let the greatest common divisor of x and y be t, so that $x = at$, $y = bt$ and (a, b) is a coprime pair with $a > b$. Since $3a^2t^2 + at = 4b^2t^2 + bt$, $a - b = t(4b^2 - 3a^2)$. Therefore, $a - b$ is divisible by t, so that $a - b = st$ for some natural number s.

Since $s = 4b^2 - 3a^2 = 4(b^2 - a^2) + a^2 = a^2 + 4st(a+b)$, a^2 is divisible by s. Therefore, $b^2 = a^2 - (a^2 - b^2)$ is divisible by s. However, (a^2, b^2) is a coprime pair, whence $s = 14$ and so $t = a - b$, $x = a(a-b)$ and $y = b(a-b)$. Therefore,

$$x - y = a(a-b) - b(a-b) = (a-b)^2 \ ,$$

a perfect square, as desired.

To find a nontrivial solution, it is enough to find a solution (a, b) of the pellian equation $(2b)^2 - 3a^2 = 1$. One such solution, which can be found by trial and error, is $(a, b) = (15, 13)$, which leads to $(x, y) = (30, 26)$.

Solution 2. We have that

$$\begin{aligned} 0 &= (3x^2 + x) - (4y^2 + y) \\ &= 3(x^2 - y^2) + (x - y) - y^2 \\ &= 4(x^2 - y^2) + (x - y) - x^2 \ , \end{aligned}$$

whence

$$x^2 = (x - y)[4(x + y) + 1]$$

and

$$y^2 = (x - y)[3(x + y) + 1] \ .$$

Multiplying these two equations yields that

$$(xy)^2 = (x - y)^2[4(x + y) + 1][3(x + y) + 1] \ .$$

The greatest common divisor of $4(x + y) + 1$ and $3(x + y) + 1$ must divide their difference $x + y$, and hence must divide 1. Therefore $4(x+y)+1$ and $3(x+y)+1$ are coprime. Since their product times a square is square, each must be a square. But then, since $(x - y)[4(x + y) + 1]$ is square, so must also be $x - y$. The result follows.

To find a solution, we observe that $3x^2 + x = 4y^2 + y$ is equivalent to

$$(8y + 1)^2 = 48x^2 + 16x + 1 \ .$$

Since $48x^2 + 16x + 1$ is a square less than $(7x)^2$, let us consider the possibility that it might be equal to $(7x - 1)^2$. The condition

$$48x^2 + 16x + 1 = (7x - 1)^2 = 49x^2 - 14x + 1$$

leads to $x = 30$. Plugging this into the original equation leads to

$$0 = 4y^2 + y - 2730 = (y - 26)(4y + 25)$$

and so $y = 26$. Thus $(x, y) = (30, 26)$ satisfies the equation.

Comment. We can use the theory of Pell's equation to generate a whole raft of solutions. Manipulating the given equation leads to

$$(8y + 1)^2 = 16(3x^2 + x) + 1 \Longrightarrow 12(8y + 1)^2 = 16(3x^2 + 12x) + 12 = 16(6x + 1)^2 - 4 \ .$$

Thus, we have to solve for positive integers the system of equations

$$u^2 - 3v^2 = 4 \ ;$$

$$u = 4(6x + 1) \ ;$$

$$v = 2(8y + 1) \ .$$

It can be shown, working modulo 4, that there are no solutions for which u and v are odd. Accordingly, $(u, v) = (2r, 2s)$ where (r, s) satisfies the pellian equation $r^2 - 3s^2 = 1$. The solutions of this equation are given by $(r,s) = (r_k, s_k)$ where $r_k + \sqrt{3}s_k = (2+\sqrt{3})^k$. The first few solutions are given by $(r,s) = (2,1), (7,4), (26,15), (97,56), (362,209), \cdots$. Not all of these will yield integers x and y. However, $(r,s) = (362,209)$ leads to $(u,v) = (724,418)$ and $(x,y) = (30,26)$. Show how you can extend this to determine an infinite family of solutions (x,y).

461. Suppose that x and y are integers for which $x^2 + y^2 \neq 0$. Determine the minimum value of the function

$$f(x,y) \equiv |5x^2 + 11xy - 5y^2| \ .$$

Solution. Say that $f(x,y)$ *represents* n if it assumes the value n for some pair (x,y) of integers not both zero. We make a number of observations. Note that $f(x,y) = f(-y,x)$, so that it is enough to look at nonnegative values of x and y. Since

$$5 = f(1,0) = f(5,-2) = f(13,-5) = \cdots \ ,$$

the number 5 is representable. Since the discriminant of the quadratic $5x^2 + 11xy - 5y^2$, namely 221, is not square, 0 is not representable by $f(x,y)$.

Suppose $p = 2$ or $p = 3$. If both x and y are multiples of p, then $f(x,y)$ is divisible by p^2, and so does not represent p. If both x and y are not multiples of p, then $x^2 \equiv y^2 \equiv 1 \pmod{p}$ and so $f(x,y) \equiv |11xy| \not\equiv 0 \pmod{p}$. Finally, if exactly one of x and y is a multiple of p, then $f(x,y)$ does not represent p. Hence, $f(x,y)$ never represents either 2 or 3.

If either or both of x and y is odd, then $f(x,y)$ must be odd. Therefore, if $f(x,y) = 4$, then $(x,y) = (2u,2v)$ for some integer pair (u,v) and $f(u,v) = 1$. It can be deduced that 4 is representable if and only if 1 is representable. Thus, the minimum representable value of f is either 1 or 5. Thus, we need to check whether the equation $5x^2 + 11xy - 5y^2 = \pm 1$ is solvable in integers.

If $f(x,y) = 1$, then $5y^2 + 11xy - 5y^2 = \pm 1$, whence $(10x + 11y)^2 - 221y^2 = \pm 20$. Let $z = 10x + 11y$. Since 221 is divisible by 13, $z^2 \equiv \pm 7 \pmod{13}$. Raising each side to the sixth power and taking account of the little Fermat theorem (that $a^{p-1} \equiv 1 \pmod{p}$ for any prime p and a not divisible by p), we find that $1 \equiv 7^6 \equiv (-3)^3 = -27 \equiv -1 \pmod{13}$, a contradiction. Hence $f(x,y)$ cannot assume the value 1.

462. For any positive real numbers a, b, c, d, establish the inequality

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+d}} + \sqrt{\frac{c}{d+a}} + \sqrt{\frac{d}{a+b}} > 2 \ .$$

Solution 1. From the arithemetic-geometric means inequality, we have that

$$\sqrt{\frac{b+c}{a}} \leq \frac{1 + (b+c)/a}{2} = \frac{a+b+c}{2a} \ ,$$

which implies that

$$\sqrt{\frac{a}{b+c}} \geq \frac{2a}{a+b+c} > \frac{2a}{a+b+c+d} \ .$$

Applying an analogous inequality to the other terms of the left side, we obtain the desired result.

Solution 2. By the arithmetic-geometric means inequality, we have that

$$\begin{aligned}\sqrt{\frac{a}{b+c}} + \sqrt{\frac{c}{d+a}} &= \frac{\sqrt{a^2+ad} + \sqrt{bc+c^2}}{\sqrt{(b+c)(a+d)}} \\ &\geq \frac{2(\sqrt{a^2+ad} + \sqrt{bc+c^2})}{a+b+c+d}\end{aligned}$$

and, similarly, that
$$\sqrt{\frac{b}{c+d}}+\sqrt{\frac{d}{a+b}}\geq\frac{2(\sqrt{ab+b^2}+\sqrt{cd+d^2})}{a+b+c+d}\ .$$
Hence the left side of the inequality is not less than $2t/s$ where $t=\sqrt{a^2+ad}+\sqrt{b^2+ab}+\sqrt{c^2+bc}+\sqrt{d^2+cd}$ and $s=a+b+c=d$.

We observe that
$$\begin{aligned}t^2&=(a^2+b^2+c^2+d^2)+(ad+ab+bc+cd)\\&\quad+2(\sqrt{a^2+ad}\sqrt{b^2+ab}+\sqrt{a^2+ad}\sqrt{c^2+bc}+\sqrt{a^2+ad}\sqrt{d^2+cd}\\&\quad+\sqrt{b^2+ab}\sqrt{c^2+bc}+\sqrt{b^2+ab}\sqrt{d^2+cd}+\sqrt{c^2+bc}\sqrt{d^2+cd})\\&>(a^2+b^2+c^2+d^2)+(ad+ab+bc+cd)+2(ab+ac+ad+bc+bd+cd)\\&>(a+b+c+d)^2=s^2\ .\end{aligned}$$
Hence $t>s$, so that $2t/s>2$ and the result follows.

Comment. The special cases are of interest. If $b=d=0$, the inequality becomes $\sqrt{a/c}+\sqrt{c/a}>2$, and the left side can be made arbitrarily close to 2 by making a close to c. If we take $d=0$, we can parameterize
$$(a,b,c)=(x^2,r^2\sin^2\theta,r^2\cos^2\theta)$$
and find that
$$\begin{aligned}\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}&=\frac{x}{2}+\tan\theta+\frac{r\cos\theta}{x}\\&\geq\tan\theta+2\sqrt{\cos\theta}\ .\end{aligned}$$
Set $f(\theta)=\tan\theta+2(\cos\theta)^{1/2}$; then $f'(\theta)=(\cos\theta)^{-1/2}(\sec^{3/2}\theta-\sin\theta)>0$, so $f(\theta)$ assumes its minimumn value of 2 when $\theta=0$.

Some other special cases have interesting denouements. If $a=c$ and $b=d$, we get
$$\sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{a+b}}+\sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{a+b}}=2\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a+b}}\right)>2\ .$$
If $a=b$ and $c=d$, we get
$$\begin{aligned}&\sqrt{\frac{a}{a+c}}+\sqrt{\frac{a}{2c}}+\sqrt{\frac{c}{a+c}}+\sqrt{\frac{c}{2a}}\\&\qquad=\frac{\sqrt{a}+\sqrt{c}}{\sqrt{a+c}}+\frac{1}{\sqrt{2}}\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right)>1+\frac{2}{\sqrt{2}}=1+\sqrt{2}>2\ .\end{aligned}$$
If you get stuck, it is often not a bad idea to try some special cases to see whether any light might be shed on how to approach the general case. In this case, the special cases were not particularly helpful, but on a contest, might help to garner a couple of marks that might not otherwise be had.

463. In *Squareland*, a newly-created country in the shape of a square with side length of 1000 km, there are 51 cities. The country can afford to build at most 11000 km of roads. Is it always possible, within this limit, to design a road map that provides a connection between any two cities in the country?

Solution. [Yifan Wang] It is possible to design a road map that provides a connection between any two cities so that the total length of the roads does not exceed 11000 km. We will provide an example.

Place the square map of the country on the Cartesian plane so that the corners are at the origin $(0,0)$ and the points $(1000,0)$, $(1000,1000)$ and $(0,1000)$. Build six main roads as follows:

R_1: A vertical road, a segment between the points $(100, 0)$ and $(100, 1000)$;
R_2: A horizontal road, a segment between the points $(100, 100)$ and $(1000, 100)$;
R_3: A horizontal road, a segment between the points $(100, 300)$ and $(1000, 300)$;
R_4: A horizontal road, a segment between the points $(100, 500)$ and $(1000, 500)$;
R_5: A horizontal road, a segment between the points $(100, 700)$ and $(1000, 700)$;
R_6: A horizontal road, a segment between the points $(100, 900)$ and $(1000, 900)$.

The six main roads are connected and have a total length of $1000 + 900 \times 5 = 5500$ km. Any of the 51 cities is at most 100 km away from one of the main roads, so the local roads that connect the cities to the main road can be built with a length of at most $51 \times 100 = 5100$ km. Thus, the length of all the roads will not exceed $5500 + 5100 = 10600 < 11000$ km.

464. A square is partitioned into non-overlapping rectangles. Consider the circumcircles of all the rectangles. Prove that, if the sum of the areas of all these circles is equal to the area of the circumcircle of the square, then all the rectangles must be squares, too.

Solution. Let s be the side length of the square and (a_i, b_i) be the dimensions of the ith rectangle. Then $s^2 = \sum a_i b_i$.

The circumcircle of the square has area $(\pi s^2)/2$ and the circumcircle of the ith rectangle has area $(\pi(a^2 + b^2))/4$. Hence, we have, using the arithmetic-geometric means inequality and the condition that the sum of the rectangular circumcircle areas is equal to the square circumcircle area,

$$\frac{\pi}{2} s^2 = \frac{\pi}{2} \sum a_i b_i \leq \frac{\pi}{4} \sum (a_i^2 + b_i^2) = \frac{\pi}{2} s^2 \ .$$

Since the extreme members of this inequality are equal, we must have equality everywhere. In particular, $a_i = b_i$ for each i and all of the partitioning rectangles are square.