Solutions for March

486. Determine all quintuplets (a,b,c,d,u) of nonzero integers for which

$$\frac{a}{b} = \frac{c}{d} = \frac{ab+u}{cd+u} \ .$$

Solution 1. If $a = b$ and $c = d$, then an integer u exists as in the problem if and only if $|a| = |c|$. Suppose that none of the fractions is equal to 1.

Let $a/b = c/d = r/s$ where the greatest common divisor of r and s is 1. Then there are integers v and w for which $(a,b) = (vr, vs)$ and $(c,d) = (wr, ws)$. If there exists a number u satisfying the conditions of the problem, then

$$\frac{r}{s} = \frac{v^2rs+u}{w^2rs+u} \Leftrightarrow rs(w^2r - v^2s) = (s-r)u \ .$$

Since the greatest common divisor of r and s is 1, it follows that gcd $(rs, r-s) = 1$ so that $s-r$ must divide

$$w^2r - v^2s = (w^2 - v^2)r - v^2(s-r) \ ,$$

and so must divide $w^2 - v^2$.

For the converse, suppose that r and s are chosen arbitrarily, and that v and w are chosen to satisfy $w^2 - v^2 = (s-r)n$ for some integer n. Let $(a,b,c,d) = (vr, vs, wr, ws)$. Then $w^2r - v^2s = (s-r)(rn - v^2)$. Let $u = rs(rn - v^2)$. Thence,

$$\begin{aligned}\frac{ab+u}{cd+v} &= \frac{v^2rs + rs(rn-v^2)}{w^2rs + rs(rn-v^2)} = \frac{v^2 + rn - v^2}{w^2 + rn - v^2} \\ &= \frac{rn}{(s-r)n + rn} = \frac{rn}{sn} = \frac{r}{s} \ ,\end{aligned}$$

as desired.

Hence (a,b,c,d,u) satisfies the given condition if and only if $a:c = b:d = v:w$, $a:b = c:d = r:s$ where $w^2 - v^2 \equiv 0 \pmod{s-r}$ and $u = rs(rn - v^2)$ for some integer n.

Solution 2. Let $a/b = c/d = r/s$ where gcd $(r,s) = 1$. Let $(a,b) = (vr, vs)$ and $(c,d) = (wr, ws)$. Then

$$\frac{r}{s} = \frac{v^2rs+u}{w^2rs+u} \ ,$$

so that $v^2rs + u = pr$ and $w^2rs + u = ps$ for some integer p. Since gcd $(r,s) = 1$, $u = qrs$ for some integer q, and

$$\frac{r}{s} = \frac{v^2+q}{w^2+q} \ .$$

Thus, $v^2 + q = hr$ and $w^2 + q = hs$ for some integer h and $w^2 - v^2 = h(s-r)$. We can now conclude as in the first solution.

Comment. A. Siddour discovered the particular family

$$(a,b,c,d,u) = (tr^2, trs, trs, ts^2, t^2r^2s^2) \ ,$$

for parameters r, s, t.

487. ABC is an isosceles triangle with $\angle A = 100^\circ$ and $AB = AC$. The bisector of angle B meets AC in D. Show that $BD + AD = BC$.

Solution 1. We have that $\angle ACB = \angle ABC = 40°$ and $\angle ABD = \angle DBC = 20°$. Locate E on BC so that $\angle BED = 80°$. Then $\angle BDE = 80°$ so that $BD = BE$.

Since $\angle DEC = 100°$ and $\angle DCE = 40°$, $\angle EDC = 40°$, so that $DE = EC$.

Since $\angle BAD + \angle BED = 180°$, the quadrilateral $ADEB$ is concyclic. Because BD bisects angle ABE, it bisects arc ADE and so $AD = DE$. It follows that

$$BD + AD = BE + EC = BC \ .$$

Comment. A variant on this argument involves a point F on the segment BE so that $BF = BA$ and $\angle DFE = \angle DEF = 80°$. From the congruence of triangles ABD and FBD, we get $AD = DF = DE = EC$, from which the result follows.

Solution 2. [A. Siddour] By the Law of Sines, $AD : BD = \sin 20° : \sin 100°$ and $BC : BD = \sin 60° : \sin 40°$. Hence

$$\begin{aligned} 1 + \frac{AD}{BD} &= 1 + \frac{\sin 20°}{\sin 100°} = \frac{\sin 100° + \sin 20°}{\sin 80°} \\ &= \frac{2 \sin 60° \cos 40°}{2 \sin 40° \cos 40°} = \frac{\sin 60°}{\sin 40°} = \frac{BC}{CD} \ , \end{aligned}$$

from which the result follows.

488. A host is expecting a number of children, which is either 7 or 11. She has 77 marbles as gifts, and distributes them into n bags in such a way that whether 7 or 11 children come, each will receive a number of bags so that all 77 marbles will be shared equally among the children. What is the minimum value of n?

Solution: 17 bags will suffice. A systematic way to approach the problem is to number the marbles and line them up in order. For a distribution to 11 children, place a marker after every seventh one (numbers 7, 14, *etc.*); for a distribution to 7 children, place a marker after every eleventh one (number 11, 22, *etc.*). This will require $6 + 10 = 16$ markers. This partitions the marbles into seventeen groups, that can be bagged accordingly. With $a \times b$ representing a bags with b marbles, we have 5×7, 2×6, 2×5, 2×4, 2×3, 2×2 and 2×1 for a distribution of

$$7, 4 + 3, 7, 1 + 6, 5 + 2, 7, 2 + 5, 6 + 1, 7, 3 + 4, 7$$

to eleven children and

$$7 + 4, 3 + 7 + 1, 6 + 5, 2 + 7 + 2, 5 + 6, 1 + 7 + 3, 4 + 7$$

to seven children.

Alternative distributions are 6×7, 5×4, 4×3, 1×2, 1×1 for distributions of

$$7, 7, 7, 7, 7, 7, 4 + 3, 4 + 3, 4 + 3, 4 + 3, 4 + 2, +1$$

and

$$7 + 4, 7 + 4, 7 + 4, 7 + 4, 7 + 4, 7 + 3 + 1, 3 + 3 + 3 + 2 \ ,$$

and

$$7 \times 7, 4 \times 4, 3 \times 3, 3 \times 1$$

for distributions of

$$7, 7, 7, 7, 7, 7, 7, 4 + 3, 4 + 3, 4 + 3, 4 + 1 + 1 + 1$$

and

$$7 + 4, 7 + 4, 7 + 4, 7 + 4, 7 + 3 + 1, 7 + 3 + 1, 7 + 3 + 1 \ .$$

Solution 1: Fewer than 17 bags will not suffice. [B. Wu] Construct a graph whose vertices consist of a set $X \cup Y$ and edges consist of a set E, where $\#X = 7$, $\#Y = 11$, representing the children if, respectively, seven and eleven show up, and the only edges join pairs (x, y) with $x \in X$ and $y \in Y$ when a bag assigned to x in one distribution goes to y in the other distribution.

This graph is connected (*i.e.*, one can construct a path of edges that is incident with every vertex). If a connected component contains m vertices of X and n vertices of Y, then the number of marbles represented by these vertices is $11m = 7n$. Since 11 and 7 are coprime, $(m, n) = (7, 11)$ and each connected component must contain all of the vertices. Since each connected graph with 18 vertices must contain at least 17 edges, at least 17 bags are necessary.

Solution 2: Fewer than 17 bags will not suffice. Since we must accommodate 11 children, no bag can contain more than 7 marbles. There cannot be more than seven bags with 7 marbles; otherwise, one of the seven children must get two bags of 7 and so more than 11 marbles. Let a *large bag* contain 4 to 6 marbles and a *small bag* contain 1 to 3 marbles.

If there are five or fewer bags with 7 marbles, then at least six of eleven children each get two bags, and we will need at least $5 + 6 \times 2 = 17$ bags. Suppose that there are six bags with 7 marbles. Then six of eleven children get 1 bag with 7 marbles and the other five get at least two bags. If any of the five get more than two bags, then we need at least 17 bags.

Suppose, if possible, that exactly six of eleven children get 1 bag with seven marbles and the other five get exactly two bags each. Then each of these five must get a large bag and a small bag. Thus, there are six bags with 7 marbles, five large and five small bags. Let us consider how they might be distributed among seven children. Six of the seven get a bag with 7 marbles and the seventh can get no more than two large bags. Thus, at least three large bags go to children who have a bag of seven marbles. If only three large bags go to such children, then the other three children with bags of seven marbles must have at least two small bags each, requiring six small bags. This is not possible. If four large bags go to children with bags of seven marbles, then the other children must each get three bags requiring at least $4 \times 2 + 3 \times 3 = 17$ bags. Finally, if five large bags go to children with a bag of seven marbles, then these five have two bags each, the sixth has at least three bags and the seventh, with only small bags, needs four bags, for a total of at least seventeen bags.

Finally, suppose that there are seven bags with 7 marbles. Then each of seven children would get one, and no remaining bag could hold more than 4 marbles. For a distribution to eleven children, seven children get a bag with 7 marbles and the remaining four get at most one bag with 4 marbles. Hence, there are at most four bags with 4 marbles. Thus, at most four of seven children get 2 bags and the remaining three get at least three bags, for a total of at least seventeen bags.

Solution 3: Fewer than 17 bags will not suffice. If, in the distribution to eleven children, five or fewer get a 7-marble bag, then we need at least $5 + 6 \times 2 = 17$ bags. If, in the distribution to seven children, four or fewer get two bags, then we need at least $4 \times 2 + 3 \times 3 = 17$ bags. So we can, henceforth, restrict attention to the situations where there are at least six 7-marble bags and at least five of seven children get two bags. Observe that, for a distribution to eleven children, all the 7-marble and 4-marble bags must go to distinct children, so that the number of 7-marble plus the number of 4-marble bags cannot exceed 10. Since there are at least six 7-marble bags and at least five of seven children get two bags, there must be at least four 4-marble bags. There cannot be seven 7-marble bags, since there would not be enough 4-marble bags available to give five of seven children two bags. Thus, we have left to consider the following situations:

Case (i): 6 7-marble bags and 5 4-marble bags. In the distribution to eleven children, we need at least $6 \times 1 + 5 \times 2 = 16$ bags, with equality if and only if we have 5 3-marble bags as well. But then there would be no combination of bags to go with one of the 7-marble bags to make a presentation of eleven marbles to one of seven children. Thus, we need at least seventeen bags.

Case(ii): 6 7-marble bags and 4 4-marble bags. Once again we need at least 16 bags to distribute to eleven children, with equality if and only if six children get one bag and five children get two. The sixteen bags must include 6 7-marble, 4 4-marble and 4 3-marble bags and, either, a 5-marble and 2-marble bag, or, a 6-marble and 1-marble bag. In either case, we cannot make a distribution to seven children. Thus, we need at least seventeen bags.

489. Suppose n is a positive integer not less than 2 and that $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq 0$,

$$\sum_{i=1}^{n} x_i \leq 400 \qquad \text{and} \qquad \sum_{i=1}^{n} x_i^2 \geq 10^4 \ .$$

Prove that $\sqrt{x_1} + \sqrt{x_2} \geq 10$. is it possible to have equality throughout? [*Bonus:* Formulate and prove a generalization.]

Solution. If $x_1 \geq 100$, the result holds trivially. Suppose that $x_1 < 100$. Then

$$\begin{aligned} 10^4 &\leq x_1^2 + \sum_{j=2}^{n} x_j^2 \leq x_1^2 + x_2 \sum_{j=2}^{n} x_j \\ &\leq x_1^2 + x^2(400 - x_1) = x_1(x_1 - x_2) + 400x_2 \\ &< 100(x_1 - x_2) + 400x_2 = 100x_1 + 300x_2 \ . \end{aligned}$$

Hence $x_1 + 3x_2 \geq 100$.

Therefore

$$\begin{aligned} (\sqrt{x_1} + \sqrt{x_2})^2 &= x_1 + x_2 + 2\sqrt{x_1 x_2} \geq x_1 + x_2 + 2x_2 \\ &= x_1 + 3x_2 \geq 100 \ , \end{aligned}$$

so that $\sqrt{x_1} + \sqrt{x_2} \geq 10$. Equality holds, for example, when $n = 16$ and $x_1 = x_2 = \cdots = x_{16} = 25$.

490. (a) Let a, b, c be real numbers. Prove that

$$\min\ [(a-b)^2, (b-c)^2, (c-a)^2] \leq \frac{1}{2}[a^2 + b^2 + c^2] \ .$$

(b) Does there exist a number k for which

$$\min\ [(a-b)^2, (a-c)^2, (a-d)^2, (b-c)^2, (b-d)^2, (c-d)^2] \leq k[a^2 + b^2 + c^2 + d^2]$$

for any real numbers a, b, c, d? If so, determine the smallest such k.
[*Bonus:* Determine if there is a generalization.]

Solution 1. (a) Wolog, $a \leq b \leq c$. Let $b - a = x \geq 0$, $c - b = y \geq 0$ and m be the minimum of x and y. Then

$$\begin{aligned} \frac{1}{2}[a^2 + b^2 + c^2] &= \frac{1}{2}[a^2 + (a+x)^2 + (a+x+y)^2] \\ &= \frac{1}{2}[a^2 + 2(a+x)^2 + 2(a+x)y + y^2] \\ &= \frac{1}{2}[(a+y)^2 + 2(a+x)^2 + 2xy] \geq xy \geq m^2 \ , \end{aligned}$$

with equality if and only if $x = y = -a$. Since m^2 is equal to the left member of the inequality, the result follows.

(b) Suppose that $a \leq b \leq c \leq d$ and let m be the minimum of the nonnegative quantities $b - a$, $c - b$, $d - c$. The m^2 is the value of the left member of the inequality.

Now,

$$\begin{aligned} 20m^2 &= 3m^2 + 2(2m)^2 + (3m)^2 \\ &\leq [(b-a)^2 + (c-b)^2 + (d-c)^2] + [(c-a)^2 + (d-b)^2] + [(d-a)^2] \\ &= 3(a^2 + b^2 + c^2 + d^2) - 2(ab + ac + ad + bc + bd + cd) \\ &= 4(a^2 + b^2 + c^2 + d^2) - (a+b+c+d)^2 \leq 4(a^2 + b^2 + c^2 + d^2) \ , \end{aligned}$$

so that $m^2 \leq \frac{1}{5}(a^2+b^2+c^2+d^2)$. Equality occurs if and only if $b-a=c-b=d-c$ and $a+b+c+d=0$. This occurs, for example, when $(a,b,c,d)=(-3,-1,1,3)$.

Solution 2. (a) Observe that

$$\begin{aligned} a^2+b^2+c^2 &= \frac{1}{3}[(a-b)^2+(b-c)^2+(c-a)^2+(a+b+c)^2 \\ &\geq \frac{1}{3}[(a-b)^2+(b-c)^2+(c-a)^2] \ . \end{aligned}$$

Wolog, assume that $a \leq b \leq c$ and note that

$$2 \text{ min } \{(b-a),(c-b)\} \leq (b-a)+(c-b)=(c-a) \ ,$$

so that

$$\begin{aligned} (a-b)^2+(b-c)^2+(c-a)^2 &\geq 2 \text{ min } \{(b-a)^2,(c-b)^2\} + 4 \text{ min } \{(b-a)^2,(c-b)^2 \\ &= 6 \text{ min } \{(b-a)^2,(c-b)^2 \ . \end{aligned}$$

The desired result follows.

(b) Wolog, let $a \leq b \leq c \leq d$ and supppose that $m = \text{min } (d-c, c-b, b-a)$. Then

$$\begin{aligned} a^2+b^2+c^2+d^2 &= \frac{1}{4}[(d-c)^2+(c-b)^2+(b-a)^2+(d-b)^2+(c-a)^2+(d-a)^2+(a+b+c+d)^2] \\ &\geq \frac{1}{4}[3m^2+2(2m)^2+(3m)^2+0] = 5m^2 \ , \end{aligned}$$

from which the result follows. Equality occurs if and only if $d-c=c-b=b-a$ and $a+b+c+d=0$.

Generalization. [B. Wu] Let $n \geq 3$ and suppose that $a_1 \leq a_2 \leq \cdots \leq a_n$. Let

$$x = a_1^2 + a_2^2 + \cdots + a_n^2$$

and

$$y = \text{min } \{(a_i - a_j)^2 : 1 \leq i < j \leq n\} \ .$$

If $i < j$, then

$$(a_j - a_i)^2 = [(a_j - a_{j-1}) + \cdots + (a_{i+1} - a_i)]^2 \geq (j-i)^2 y \ ,$$

whence

$$\sum_{1 \leq i < j \leq n} (j-i)^2 y \leq \sum_{1 \leq i < j \leq n} (a_j - a_i)^2 = n \sum_{i=1}^{n} a_i^2 - (\sum_{i=1}^{n} a_i)^2 \leq nx \ .$$

Therefore,

$$\begin{aligned} \frac{x}{y} &\geq \frac{1}{n} \sum_{1 \leq i < j \leq n} (j-i)^2 = \frac{1}{n}\left[n \sum_{i=1}^{n} i^2 - (\sum_{i=1}^{n} i)^2 \right] \\ &= \frac{1}{n}\left[\frac{n^2(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4} \right] \\ &= \frac{n(n+1)}{12}[2(2n+1) - 3(n+1)] = \frac{n(n+1)}{12}(n-1) = \frac{n(n^2-1)}{12} \ . \end{aligned}$$

Hence

$$\text{min } \{(a_i - a_j)^2 : 1 \leq i, j \leq n\} \leq \frac{12}{n(n^2-1)} \sum_{i=1}^{n} a_i^2 \ .$$

This yields the coefficients $\frac{1}{2}$ and $\frac{1}{5}$ for $n = 3$ and $n = 4$, respectively. For equality, we require that $a_{i+1} - a_i$ are all equal and that $a_1 + a_2 + \cdots + a_n = 0$. This occurs for example when n is even and

$$(a_1, a_2, \cdots, a_n) = (-(n-1), -(n-3), \cdots, (n-3), (n-1))$$

and when n is odd and

$$(a_1, a_2, \cdots, a_n) = (-\frac{1}{2}(n-1), \frac{1}{2}(n-3), \cdots, -1, 0, 1, \cdots \frac{1}{2}(n-3), \frac{1}{2}(n-1)) \ .$$

491. Given that x and y are positive real numbers for which $x + y = 1$ and that m and n are positive integers exceeding 1, prove that

$$(1 - x^m)^n + (1 - y^n)^m > 1 \ .$$

Solution 1. (Probability) Suppose that 0's and 1's are to be placed at random in the slots of an $m \times n$ rectangular array. Let x be the probability that 1 goes in any given slot and y the probability that 0 goes into the slot. Then x^m is the probability that all the m slots in a given column receive 1 and $1 - x^m$ the probability that at least one slot in the column receives 0. Thus, $(1 - x^m)^n$ is the probability that at least one slot in every column receives 0. Similarly, $(1 - y^n)^m$ is the probability that at least one slot in every row receives 1.

It is possible for both of these events to occur. [Explain how to do this.] However, also, at least one of these events must occur. For, suppose there is a column that contains no 0; then all of its entries must be 1 and so each row contains at least one 1.

Since these two events exhaust all possibilities and are not mutually exclusive, the sum of their probabilities exceeds one:

$$(1 - x^m)^n + (1 - y^n)^m > 1 \ .$$

Solution 2. [A. Logan] Observe first that, if f is a convex real function and if $p \leq q \leq r \leq s$ and $p + s = q + r$, then

$$f(q) + f(r) \leq f(p) + f(s) \ .$$

To see this, add the inequalities

$$f(q) \leq \left(\frac{s-q}{s-p}\right) f(p) + \left(\frac{q-p}{s-p}\right) f(s)$$

and

$$f(r) \leq \left(\frac{s-r}{s-p}\right) f(p) + \left(\frac{r-p}{s-p}\right) f(s) \ .$$

If f is strictly convex, then the inequalities are strict.

Consider the substitution $f(t) = t^n$ (for $t \geq 0$), which is strictly convex, $p = y(1 - x^m)$, $q = 1 - x = y$, $r = 1 - x^m$, $s = 1 - x^{m+1}$. Then, for $m, n \geq 1$,

$$y^n + (1 - x^m)^n \leq y^n (1 - x^m)^n + (1 - x^{m+1})^n \ ,$$

with strict inequality when $n \geq 2$.

Fix $n \geq 2$. When $m = 1$, we have that

$$(1 - x^m)^n + (1 - y^n)^m = y^n + 1 - y^n = 1 \ .$$

We prove that the inequality of the problem holds for each $m > 1$ by induction on m. Assuming it for m, we have that

$$\begin{aligned}(1 - y^n)^{m+1} &= (1 - y^n)^m (1 - y^n) \geq [1 - (1 - x^m)^n](1 - y^n) \\ &= 1 - y^n - (1 - x^m)^n + y^n (1 - x^m)^n > 1 - (1 - x^{m+1})^n \ ,\end{aligned}$$

using the induction hypothesis and the result of the previous paragraph.

Solution 3. (Calculus) Let

$$\phi(x) = (1 - x^m)^n + [1 - (1-x)^n]^m - 1 \; .$$

Then

$$\begin{aligned}\phi'(x) &= n(1-x^m)^{n-1}(-mx^{m-1}) + m[1-(1-x^n)]^{m-1}n(1-x)^{n-1} \\ &= -n(1-x)^{n-1}(1+x+\cdots+x^{m-1})^{n-1}mx^{m-1} \\ &\qquad\qquad + mx^{m-1}[1+(1-x)+\cdots+(1-x)^{n-1}]^{m-1}n(1-x)^{n-1} \\ &= mn(1-x)^{n-1}x^{m-1}[f(x)-g(x)] \; ,\end{aligned}$$

where

$$f(x) = [1 + (1-x) + \cdots + (1-x)^{n-1}]^{m-1}$$

and

$$g(x) = [1 + x + \cdots + x^{m-1}]^{n-1} \; .$$

The function $f(x)$ decreases from $f(0) = n^{m-1}$ to $f(1) = 1$ and $g(x)$ increases from $g(0) = 1$ to $g(1) = m^{n-1}$ for $0 \leq x \leq 1$. Hence $f(x) - g(x)$ is a decreasing function from $f(0) - g(0) = n^{m-1} - 1 > 0$ to $f(1) - g(1) = 1 - m^{n-1} < 0$ Therefore, there exists a number c with $0 < c < 1$ for which $\phi'(x) > 0$ for $0 < x < c$, $\phi'(c) = 0$ and $\phi'(x) < 0$ for $c < x < 1$. Hence $\phi(x)$ increases for $0 < x < c$ and decreases for $c < x < 1$. Since $\phi(0) = \phi(1) = 0$, it follows that $\phi(x) > 0$ as desired.

Comment. If $m = n > 1$ and $x = y = \frac{1}{2}$, the inequality becomes $2(1 - 2^{-n})^n > 1$, which reduces to the intriguing

$$\frac{1}{2^n} + \frac{1}{2^{1/n}} < 1 \; .$$

This can be shown directly by noting that $n \leq 2^{n-1}$ for $n \geq 2$ and

$$\left(1 - \frac{1}{2^n}\right)^n > 1 - \frac{n}{2^n} \geq \frac{1}{2} \; .$$

492. The faces of a tetrahedron are formed by four congruent triangles. if α is the angle between a pair of opposite edges of the tetrahedron, show that

$$\cos\alpha = \frac{\sin(B-C)}{\sin(B+C)}$$

where B and C are the angles adjacent to one of these edges in a face of the tetrahedron.

Solution 1. Let the vertices of the tetrahedron by $ABCD$. We must have $|BC| = |AD| = a$, , $|AC = |BD = b$, $|AB| = |CD| = c$. Let the opposite edges in question be BC and AD, both of length a, and let the angle between $\overrightarrow{BD}$ and $\overrightarrow{BC}$ be B and the angle between $\overrightarrow{CD}$ and $\overrightarrow{CB}$ be C. These are two angles of each triangular face, the remaining angle being $180° - (B + C)$ whose sine is the same as that of $B + C$. By the Law of Sines, we have that $a \sin C = b \sin(B + C)$ and $a \sin B = c \sin(B + C)$.

Then

$$\begin{aligned} a^2 \cos\alpha &= \overrightarrow{BC} \cdot \overrightarrow{DA} = (\overrightarrow{DC} - \overrightarrow{DB}) \cdot \overrightarrow{DA} \\ &= \overrightarrow{DC} \cdot \overrightarrow{DA} - \overrightarrow{DB} \cdot \overrightarrow{DA} = ca\cos C - cb\cos B \\ &= [\sin(B+C)]^{-1}[a^2 \sin B \cos C - a^2 \sin C \cos B] = a^2[\sin(B+C)]^{-1}\sin(B-C) \; ,\end{aligned}$$

which yields the desired result.

Comment. Note that there is a sign ambiguity in the answer, depending on which ends of the edge are assigned the angles B and C; in the solution, the choice was made to make the answer come out right.

Solution 2. Complete the parallelogram $ABCE$ in the plane of triangle ABC so that $AE \| BC$ and $EC \| AB$. Suppose that AC and BE intersect at F; then F is the midpoint of both AC and BE. Let $|AC| = |BD| = b$, $|BC| = |AD| = a$, $|AB| = |CD| = c$, $|BF| = |DF| = |EF| = x$ and $|DE| = u$.

Then $2(a^2 + c^2) = b^2 + 4x^2$ and $2(u^2 + b^2) = 4x^2 + 4x^2 = 8x^2$, whence $u^2 = 2(a^2 + c^2 - b^2)$. Since $u^2 = 2a^2(1 - \cos\alpha)$,

$$\cos\alpha = 1 - \frac{u^2}{2a^2} = \frac{b^2 - c^2}{a^2} \ .$$

Now

$$\begin{aligned}\sin(B - C) &= \sin B \cos C - \sin C \cos B \\ &= \frac{b}{2R} \cdot \frac{a^2 + b^2 - c^2}{2ab} - \frac{c}{2R} \cdot \frac{a^2 + c^2 - b^2}{2ac} \\ &= \frac{2(b^2 - c^2)}{4aR} \ ,\end{aligned}$$

and, similarly, $\sin(B + C) = 2a^2/4aR$. Hence

$$\frac{\sin(B - C)}{\sin(B + C)} = \frac{b^2 - c^2}{a^2} = \cos\alpha \ .$$