Solutions for September.

514. Prove that there do not exist polynomials $f(x)$ and $g(x)$ with complex coefficients for which

$$\log_b x = \frac{f(x)}{g(x)}$$

where b is any base exceeding 1.

Solution 1. Suppose that the given equation is possible. Then we must have, for each positive integer n,

$$\frac{f(x^n)}{g(x^n)} = \log_b x^n = n \log_b x = \frac{nf(x)}{g(x)} \ ,$$

so that $f(x^n)g(x) = nf(x)g(x^n)$. However, by comparing the leading coefficients of the two sides, we see that this is impossible.

Solution 2. We presume that the relation is to be an identity for all x for which both sides are defined, in particular when x is a positive integer. Since $\log_b x = \log_2 x / \log_2 b$, $\log_b x$ is a constant multiple of $\log_2 x$. Thus, it is enough to prove the result with $b = 2$.

It is easily established by induction that, for each positive integer x, $x \leq 2^{x-1}$, so that $\log_2 x \leq x - 1$. Applying this to $x^{1/2}$, where x is a square, this yields

$$\log_2 x = 2 \log_2(x^{1/2}) \leq 2(x^{1/2} - 1) < 2x^{1/2} \ .$$

Suppose, if possible, that there exist polynomials $f(x)$ and $g(x)$ such that, for every positive integer x,

$$\log_2 x = \frac{f(x)}{g(x)} \ .$$

We may suppose that the leading coefficient of $g(x)$ is 1 and that the respective degrees of $f(x)$ and $g(x)$ are the positive integers m and n.

Suppose that

$$f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 = a_m x^m (1 + a_{m-1}x^{-1} + \cdots + a_0 x^{-m})$$

and

$$f(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 = x^n (1 + b_{n-1}x^{-1} + b_{n-2}x^{-2} + \cdots + b_0 x^{-n}) \ .$$

Since

$$|x^{-1}(a_{m-1} + \cdots + a_0 x^{1-m})| \leq x^{-1}(|a_{m-1}| + \cdots + |a_0|) \leq Mx^{-1}$$

and

$$|x^{-1}(b_{n-1} + \cdots + b_0 x^{1-n})| \leq x^{-1}(|b_{n-1}| + \cdots + |a_0|) \leq Mx^{-1}$$

where M is the maximum of $|a_{m-1}| + \cdots + |a_0|$ and $|b_{n-1}| + \cdots + |b_0|$, and we can select N such that $Mx^{-1} < 1/2$, for $x > N$, we have that

$$\frac{1}{2} < 1 - |a_{m-1}x^{-1} + \cdots + a_0 x^{-m}| \leq 1 + a_{m-1}x^{-1} + \cdots + a_0 x^{-m} < 1 + |a_{m-1}x^{-1} \cdots a_o x^{-m}| < \frac{3}{2}$$

and

$$\frac{1}{2} < 1 - |b_{n-1}x^{-1} + \cdots + b_0 x^{-m}| \leq 1 + b_{n-1}x^{-1} + \cdots + b_0 x^{-m} < 1 + |b_{m-1}x^{-1} \cdots b_0 x^{-m}| < \frac{3}{2} \ ,$$

so that

$$\frac{1}{3} < \frac{1 + a_{m-1}x^{-1} + \cdots + a_0 x^{-m}}{1 + b_{n-1}x^{-1} + \cdots + b_0 x^{-}} < 3 \ .$$

Then, for $x > 2N$, $f(x)/g(x)$ lies between $\frac{1}{3}a_m x^{m-n}$ and $3a_m x^{m-n}$, so that a_m must be positive.

If $m \leq n$, then $f(x)/g(x)$ is bounded whereas $\log_2 x$ is not. Hence $m > n$. But then for x a large square integer exceeding $2M$,

$$1 = (\log_2 x)^{-1}(f(x)/g(x)) > (1/2x^{1/2})[(1/3)a_m x^{m-n}] = (1/6)a_m x^{m-(1/2)-n} \ .$$

This is a contradiction for sufficiently large x and the result follows.

515. Let n be a fixed positive integer exceeding 1. To any choice of n real numbers x_i satisfying $0 \leq x_i \leq 1$, we can associate the sum

$$\sum\{|x_i - x_j| : 1 \leq i < j \leq n\} \ .$$

What is the maximum possible value of this sum and for which values of the x_i is it assumed?

Solution 1. Wolog, we may suppose that $1 \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$. Then the sum in question is equal to

$$\sum\{x_i - x_j : 1 \leq i < j \leq n\} = (n-1)x_1 + (n-3)x_2 + \cdots + (n+1-2i)x_i + \cdots - (n-1)x_n \ ,$$

Since $0 \leq x_i \leq 1$ for each i, this sum is dominated by $(n-1) + (n-3) + \cdots$, where the sum is taken over all the indices yielding positive coefficients and equality occurs when $x_i = 1$ for these indices.

When $n = 2m$ is even, this maximum sum is equal to $(2m-1)+(2m-3)+\cdots+3+1 = m^2 = n^2/4$; we can achieve it with $x_1 = \cdots = x_m = 1$ and $x_{m+1} = x_{m+2} = \cdots = x_{2m} = 0$. When $n = 2m+1$ is odd, this maximum sum is equal to $(2m) + (2m-2) + \cdots + 2 = 2(m + (m-1) + \cdots + 1) = m(m+1) = (n^2-1)/4$; it is achieved with $x_1 = x_2 = \cdots = x_m = 1$ and $x_{m+2} = x_{m+3} = \cdots = x_{2m+1} = 0$ (the value of x_{m+1} being immaterial).

In summary, the maximum sum can be rendered as $\lfloor n^2/4 \rfloor$.

Solution 2. We may assume that $1 \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$. For $1 \leq i \leq n-1$, let $d_i = x_i - x_{i+1}$, so that $|x_i - x_j| = x_i - x_j = d_i + d_{i+1} + \cdots + d_{j-1}$ when $i < j$. Suppose $1 \leq p \leq n$ is chosen so that $k(n-k) \leq p(n-p)$ for each $1 \leq k \leq n$.

Note that, in the given sum, each d_k occurs in the expansion of terms of the form $x_i - x_j$ where $1 \leq i \leq k$ and $k+1 \leq j \leq n$; there are $k(n-k)$ such terms. Therefore

$$\begin{aligned}\sum\{|x_i - x_j| : 1 \leq i < j \leq n\} &= \sum_{k=1}^{n-1} k(n-k)d_k \\ &\leq p(n-p)\sum_{k=1}^{n-1} d_k \leq p(n-p) \ .\end{aligned}$$

with equality occurring if $d_1 = d_2 = \cdots = d_{p-1} = d_{p+1} = \cdots = d_{n-1} = 0$ and $d_p = x_p - x_{p+1} = 1$, *i.e.* $x_1 = \cdots = x_p = 1$ and $x_{p+1} = \cdots = x_n = 0$.

Since $p(n-p) - k(n-k) = (p-k)(n-p-k) \geq 0$ for all k if and only if $k \leq p \leq n-k$ or $(n-k) \leq p \leq k$ for all k, we see that $p = n/2$ when n is even and $p = (n \pm 1)/2$ when n is odd. This produces the answer in Solution 1.

516. Let $n \geq 1$. Is it true that, for any $2n+1$ positive real numbers $x_1, x_2, \cdots, x_{2n+1}$, we have that

$$\frac{x_1 x_2}{x_3} + \frac{x_2 x_3}{x_4} + \cdots + \frac{x_{2n+1} x_1}{x_2} \geq x_1 + x_2 + \cdots + x_{2n+1} \ ,$$

with equality if and only if all the x_i are equal?

Solution 1. Let $n = 1$. Then we have that

$$2\left(\frac{x_1x_2}{x_3} + \frac{x_2x_3}{x_1} + \frac{x_3x_1}{x_2}\right) = \left(\frac{x_1x_2}{x_3} + \frac{x_2x_3}{x_1}\right) + \left(\frac{x_2x_3}{x_1} + \frac{x_3x_1}{x_2}\right) + \left(\frac{x_3x_1}{x_2} + \frac{x_1x_2}{x_3}\right)$$
$$\geq 2x_1 + 2x_2 + 2x_3 = 2(x_1 + x_2 + x_3) \ ,$$

with equality if and only of $x_1 = x_2 = x_3$, by the arithmetic-geometric means inequality. Thus, the inequality holds for $n = 1$.

The inequality is not generally true for $n \geq 2$. For each positive integer n, define the function

$$L_n(x_1, x_2, \cdots, x_{2n+1}) = \frac{x_1x_2}{x_3} + \frac{x_2x_3}{x_4} + \cdots + \frac{x_{2n+1}x_1}{x_2} \ .$$

Observe that

$$L_{n+1}(x_1, x_2, \cdots, x_{2n+1}, x_1, x_2) = L_n(x_1, x_2, \cdots, x_{2n+1}) + \frac{x_1x_2}{x_1} + \frac{x_2x_1}{x_2} = L_n(x_1, x_2, \cdots, x_{2n+1}) + x_1 + x_2 \ .$$

Thus, if we can determine $(x_1, x_2, \cdots, x_{2n+1})$ to contradict the inequality, then $(x_1, x_2, \cdots, x_{2n+1}, x_1, x_2)$ contradicts the inequality at the nest higher level. Accordingly, to prove our assertion, is suffices to find a counterexample when $n = 2$.

Since

$$L_2(90, 3, 9, 1, 1) = 30 + 27 + 9 + (1/90) + 30 < 97 < 90 + 3 + 9 + 1 + 1 \ ,$$

it follows that the inequality generally fails for $n \geq 2$.

Solution 2. By the Cauchy-Schwarz Inequality, we have in the case $n = 1$,

$$\left(\frac{x_1x_2}{x_3} + \frac{x_2x_3}{x_1} + \frac{x_3x_1}{x_2}\right)\left(\frac{x_1x_3}{x_2} + \frac{x_2x_1}{x_3} + \frac{x_3x_2}{x_1}\right) \geq \left(\sqrt{\frac{x_1^2x_2x_3}{x_3x_2}} + \sqrt{\frac{x_2^2x_3x_1}{x_1x_2}} + \sqrt{\frac{x_3^2x_1x_2}{x_2x_1}}\right)^2 = (x_1 + x_2 + x_3)^2 \ .$$

from which the desired inequality follows.

Suppose that $n \geq 2$. Let $x_1 = 3^4$, $x_2 = 3$, $x_3 = 3^2$, $x_4 = x_5 = \cdots = x_{2n+1} = 1$, so that $x_1 + x_2 + \cdots + x_{2n+1} = 81 + 3 + 9 + (2n - 2) = 91 + 2n$. Then

$$\frac{x_1x_2}{x_3} = \frac{x_2x_3}{x_4} = \frac{x_{2n+1}x_1}{x_2} = 3^3 \ ,$$

$$\frac{x_3x_4}{x_1} = 3^2 \ , \qquad \frac{x_{2n}x_{2n+1}}{x_1} = \frac{1}{3^4} \ ,$$

and

$$\frac{x_ix_{i+1}}{x_{i+2}} = 1 \qquad \text{for} \ \ i = 4, \cdots, 2n - 1 \ .$$

(The last case is vacuous when $n = 2$.) The sum of the left side of the purported inequality is $3 \times 3^3 + 3^2 + (1/3^4) + (2n - 4) < 81 + 9 + 1 + (2n - 4) = 87 + 2n$. Thus, the left side is less than the right side and we have a counterexample.

Comment. The case $n = 1$ was proven by W.P. Wen and the general counterexample is adapted from one given by A. Remorov.

517. A man bought four items in a *Seven-Eleven* store. The clerk entered the four prices into a pocket calculator and *multiplied* to get a result of 7.11 dollars. When the customer objected to this procedure, the clerk realized that he should have added and redid the calculation. To his surprise, he again got the answer 7.11. What did the four items cost?

Solution. Let the cost in cents of the four items be a, b, c, d. Then a, b, c, d are whole numbers with $a + b + c + d = 711 = 3^2 \times 79$ and

$$\left(\frac{a}{100}\right)\left(\frac{b}{100}\right)\left(\frac{c}{100}\right)\left(\frac{d}{100}\right) = \frac{711}{100} \ .$$

so that $abcd = 711 \times 10^6 = 2^6 \times 3^2 \times 5^6 \times 79$. Exactly one price (in cents) is a multiple of 79, and at most three prices (in cents) are even or are a multiple of 5,

It is not possible for three prices to be a multiple of 25. Otherwise, the remaining price would be the multiple of 79, and the sum of the three remaining prices would also be a multiple of 79 as well as of 25. But $79 \times 25 > 711$, and this is not possible. Hence, at least one of the prices is a multiple of $5^3 = 125$; this price is clearly not a multiple of 79.

Case 1: One of the prices is $5 \times 79 = 395$. Suppose that $a = 5 \times 79 = 395$. Suppose that b is a multiple of $5^3 = 125$. Since not all four prices can be a multiple of 5, one price, c, say, must be a multiple of $5^2 = 25$.

If $(a, b) = (395, 125)$, then, modulo 25, $a + b + c \equiv 20$, so that $d \equiv 11 - 20 \equiv 16$. Since d can have only 2, 3, 5 as prime divisor, $d = 16$. But this leads to $c = 175 = 7 \times 5^2$, which is not possible. If $(a, b) = (395, 250)$, again $d = 16$ so that $c = 50 = 2 \times 2 \times 5^2$. But then $abcd$ is not divisible by 3. Since $a + b < 711$, this exhausts the possibilities and Case 1 cannot occur.

Case 2. One of the prices, say a is one of the multiples 79, 158, 231, 316, 474 of 79 and another, say b is one of the multiples 125, 250, 375, 500, 625 of 125. Examining the cases and conducting an analysis similar to that of Case 1, we arrive at the unique solution

$$(a, b, c, d) = (316, 125, 150, 120) = (2^2 \times 79, 5^3, 2 \times 3 \times 5^2, 2^3 \times 3 \times 5 \ .$$

Therefore the four items cost \$1.20, \$1.25, \$1.50 and \$3.16.

Comments. There are a couple of "near misses" where the product is off by a prime factor: $(45, 100, 250, 316) = (3^2 \times 5, 2^2 \times 5^2, 2 \times 5^3, 2^2 \times 79)$ and $(25, 120, 250, 316) = (5^2, 2^3 \times 3 \times 5, 2 \times 5^3, 2^2 \times 79)$.

AQ. Zhang had an interesting way to reject some cases. Suppose that $a = 395 = 5 \times 79$. Then $b + c + d = 316$ and $bcd = 2^6 \times 3^2 \times 5^5$. This gives an arithmetic mean for b, c, d less than 108 and a geometric mean that satisfies

$$\sqrt[3]{2^6 \times 3^2 \times 5^5} = 2^2 \times 5 \times \sqrt[3]{9 \times 25} = 20 \times \sqrt[3]{225} > 20 \times 6 = 120 \ .$$

This is impossible by the arithmetic-geometric means ineqaulity. Similarly, of $a = 375 = 3 \times 5^3$, the arithmetic mean of b, c, d is 112, while the geometric mean is $20 \times \sqrt[3]{237}$ which exceeds 120. Again, this is not possible.

In general $bcd = \frac{10^6 \times 711}{a}$ exceeds 10^6, so that the geometric mean of b, c, d is always at least 100. If $a > 411$, the geometric mean is less than 100. Thus, we eliminate from consideration all multiples of 79 greater than 316 and all multiples of 125 greater than 250.

518. Let I be the incentre of triangle ABC, and let AI, BI, CI, produced, intersect the circumcircle of triangle ABC at the respective points D, E, F. Prove that $EF \perp AD$.

Solution 1. Let $\alpha = \angle BAD = \angle CAD$, $\beta = \angle ABE = \angle CBE$ and $\gamma = \angle ACF = \angle FCB$. Suppose that AI and EF intersect at G. Since $\angle FAB = \angle FCB = \gamma$ and $\angle AFE = \angle ABE = \beta$, it follows that $\angle AGE = \angle FAG + \angle AFG = \alpha + \gamma + \beta = 90°$ and $EF \perp AD$.

Solution 2. Use the same notation as in Solution 1.

$$\angle IGE = \angle IFG + \angle FIG = \angle CFE + \angle AIF = \angle CBE + \angle ACF + \angle IAC = \beta + \gamma + \alpha = 90° \ .$$

Solution 3. Use the same notation as in Solution 1. A rotation with centre F and angle β carries ray FE onto FC. A rotation with centre C and angle γ carries ray CF onto CA. A rotation with centre A and angle α carries AC onto AD. Since all of these rotation have the same sense and the sum of their angles is $90°$, it follows that the final position AD of the line EF is perpendicular to EF and the result follows.

Solution 4. Let U be the intersection of AB and CF, P of AB and FE, V of AC and BE and Q of AC and FE. Since $\angle CFE = \angle CBE = \angle EBA$ and $\angle FUA = \angle BUI$, triangles BIU and FPU are similar, so that $\angle BIU = \angle FPU$. Similarly, triangles CIV and EQV are similar, and $\angle CIV = \angle EQV$. Hence, in triangles APG and AQG,

$$\angle APG = \angle FPU = \angle BIU = \angle CIV = \angle EQV = \angle AQG \ .$$

Also, $\angle PAG = \angle QAG$. Therefore, $\angle AGP = \angle AGQ$, and the result follows since P, G, Q are collinear.

519. Let AB be a diameter of a circle and X any point other than A and B on the circumference of the circle. Let t_A, t_B and t_X be the tangents to the circle at the respective points A, B and X. Suppose that AX meets t_B at Z and BX meets t_A at Y. Show that the three lines YZ, t_X and AB are either concurrent (i.e. passing through a common point) or parallel.

Solution. Let t_X intersect t_A and t_B in U and V respectively, and let O be the centre of the circle. If X is the midpoint of the arc AB, then t_X is parallel to AB and the reflection in the diameter of the circle passing through X interchanges A and B, U and V, as well as Y and Z. Hence AB, $UV = t_X$ and YZ, being perpendicular to the diameter, are all parallel.

Henceforth, suppose, say, that X is closer to A than to B. Let $\alpha = \angle XAB$ and $\beta = \angle XBA$. Then, by standard results on isoceles triangles and subtended angles, we have that

$$\alpha = \angle XAB = \angle AXO = \angle AYB = \angle UXY = \angle VBY = \angle VXB$$

and

$$\beta = \angle XBA = \angle OXB = \angle AXU = \angle UAX = \angle BZX = \angle ZXV \ ;$$

also $YU = UX = UA$ and $ZV = XV = BV$.

Thus, U and V are the respective midpoints of AY and BZ. Let BA and ZY intersect at W. Since $AY \| BZ$, the dilatation with factor $|WB|/|WA|$ and centre W takes A to B, Y to Z, and the midpoint U of AY to the midpoint V of BZ. Hence W, U and V are collinear and the result follows.

520. The *diameter* of a plane figure is the largest distance between any pair of points in the figure. Given an equilateral triangle of side 1, show how, by a stright cut, one can get two pieces that can be rearranged to form a figure with maximum diameter

(a) if the resulting figure is convex (*i.e.* the line segment joining any two of its points must lie inside the figure);

(b) if the resulting figure is not necessarily convex, but it is connected (*i.e.* any two points in the figure can be connected by a curve lying inside the figure).

Solution. (a) The maximum diameter is $\sqrt{13}/2$.

We first observe that for a convex polygon, the diameter is realized by joining some two of its vertices. To see this, let PQ be any segment contained within the figure and draw two lines l and m perpendicular to PQ through P and Q respectively. Move l in the direction $\overrightarrow{QP}$ to the last position for which it has a nonvoid intersection with the polygon; this intersection must contain a vertex U (it consists of either a side or a vertex of the polygon). Similarly, move m in the direction $\overrightarrow{PQ}$ until it contains a vertex V. Then the distance between U and V must be at least as great as the distance between the lines l and m, which is at least as great as the distance between P and Q.

In cutting the equilateral triangle ABC, there are two possibilities. Either the cut passes through a vertex A and an interior point D of the opposite side BC. Or it passes through an interior point E of a side AB and an interior point F of a side AC.

Suppose first that the cut is AD, through a vertex. For a convex result, we must place triangle ABD against triangle ADC so that one side of one triangle lies along an equal side of the other. There are generally three ways to do this.

(i) Turn ABD over so that A falls on D and D falls on A. Let B fall on U. The diameter of $ACDU$ is equal to the maximum length of the four sides and two diagonals. The lengths of four sides and of the diagonal AD do not exceed 1.

$$|CU| \leq |CA| + |AU| = |CA| + |DB|$$

and

$$|CU| \leq |CD| + |DU| = |CD| + |AB| \ .$$

Hence

$$|CU| \leq 1 + \ \min\ (|DB|, |CD|) \leq \frac{3}{2}$$

. The diameter of this figure does not exceed $3/2$.

(ii) Move triangle ABD so that A stays put, B falls on C and D goes to a point V (this is a rotation about A). Since $|DV| \leq |DC| + |CV| = |DC| + |BD|$, it can be seen that the diameter of this figure does not exceed 1.

(iii) Move triangle ABD so that A falls on C, B falls on A and D falls on W. Since $|DW|$ does not exceed the minimum of $|AD| + |AW| = |AD| + |BD|$ and $|CW| + |DC| = |AD| + |DC|$, we can deduce that the diameter does not exceed $3/2$.

In the case that D is the midpoint of BC, there are two additional possibilities.

(iv) Place triangle ABD alongside triangle ACD that they have the side CD in common to get an obtuse isosceles triangle whose longest side has length $\sqrt{3}$.

(v) Finally place triangle ABD alongside triangle ACD so that B falls on D and D falls on C to get a $150°, 30°$ parallellogram with side lengths 1 and $\sqrt{3}/2$. By the law of cosines, the length of the longer diagonal of this parallelogram is the square root of

$$1 + \frac{3}{4} - \sqrt{3}\cos 150° = 1 + \frac{3}{4} + \frac{3}{2} = \frac{13}{4} \ ,$$

so that the diameter of this figure turns out to be $\sqrt{13}/2$. Note that this exceeds $\sqrt{3}$.

Consider the second possibility in which the cut EF joins a point E in AB to a point F in AC. A side of the triangle AFE must be placed against an equal side of the quadrilateral $BCFE$. No side of triangle AFE can be placed against BC, since BC is longer than any chord of triangle ABC except sides AB and AC. The equality of FE with either AE or AF occurs exactly when $EF \| BC$ and E and F are the respective midpoints of the sides.

We might consider turning triangle AFE over so that E and F are interchanged. If one of the angles, say $\angle AFE$, exceeds the other, $\angle AEF$, then $\angle BEF + \angle AFE > \angle BEF + \angle AEF = 180°$ and we would not get a convex figure. If the two angles $\angle AEF$ and $\angle AFE$ are equal, then we get triangle ABC with diameter 1. Thus, we find that there are essentially six cases.

(i) Suppose that $|EF| = |FC| = x$ and that triangle AEF is moved so that E falls on F, F falls on C and A falls on P. Let $y = |AE| = |FP|$. This gives a pentagon $BCPFE$ whose respective side lengths are 1, $1 - x$, y, x, $1 - y$, none of which exceeds 1. The three diagonals that lie within triangle ABC have length less than 1. Since both angles EBP and EPB are less than $60°$, BEP is the largest angle of triangle BEP and so $|EP| < |BP|$. Finally,

$$|BP| < |BE| + |EF| + |FP| = (1 - y) + x + y = 1 + x$$

and

$$|BP| < |BC| + |CP| = 1 + (1 - x) = 2 - x \ ,$$

it follows that $|BP|$ is less than the minimum of $1+x$ and $2-x$, which cannot exceed $3/2$. Thus, the diameter of $BCPFE$ is less than $3/2$.

(ii) Suppose that $|EF| = |FC| = x$ and triangle AEF is moved so that F stays put, E falls on C and A falls on Q. With $y = |AE| = |QC|$, we get a pentagon $BCQFE$ with respective side lengths $1, y, 1-x, x, 1-y$. We find that

$$|BQ| < \min\ (1+y, (1-y)+x+(1-x)) = \min\ (1+y, 2-y) < 3/2$$

and that all other sides and diagonals of the pentagon do not exceed 1. Hence the diameter is less than $3/2$.

(iii) Suppose that $|AE| = |FC| = x$, so that $|AF| = 1-x$. Let triangle AEF be moved so that A falls on C, E falls on F and F falls on R. Let $y = |EF| = |FR|$. The side lengths of pentagon $BCRFE$ are respectively $1, 1-x, y, y, 1-x$. Since $|AE| > |AF|$, $x > 1/2$. We have that

$$|ER| < |BR| < |BC| + |RC| = 2 - x < 3/2$$

and so the diameter of the pentagon must be less than $3/2$.

(iv) Suppose that $|AE| = |FC| = x$ and that triangle AEF is moved so that A falls on F, E falls on C and F falls on S. Using the Law of Cosines on triangle AEF, we have that $y = \sqrt{3x^2 - 3x + 1}$. Then

$$|BS| \leq |BE| + |EF| + |FS| = 2(1-x) + \sqrt{3x^2 - 3x + 1}\ .$$

Since $x \geq 1/2$, we have that

$$(4x^2 - 2x + (1/4)) - (3x^2 - 3x + 1) = x^2 + x - (3/4) = (1/4)(2x-1)(2x+3) > 0\ .$$

Hence

$$\sqrt{3x^2 - 3x + 1} < \sqrt{4x^2 - 2x + (1/4)} = 2x - (1/2) = (3/2) - 2(1-x)\ .$$

Therefore $|BS| < 3/2$. Thus, the diameter of the figure obtained is less than $3/2$.

(v) Suppose that $|AF| = |FC|$ (so that F is the midpoint of AC) and that triangle AEF is moved so that F is left in place, A falls on C and E falls on T. Then we get a quadrilateral $BCTE$ and find that

$$|BT| \leq |BF| + |FT| \leq 2|BF| = \sqrt{3}\ .$$

Thus, the diameter does not exceed $\sqrt{3}$.

(vi) Suppose that $|AF| = |FC|$ and that triangle AEF is moved so that A falls on F, F falls on C and E falls on U. Let H be on FU produced so that $HC \perp AC$. Since $|FC| = 1/2$ and $\angle CFH = 60°$, $BCHF$ is a $150°$, $30°$ parallelogram with side lengths 1 and $\sqrt{3}/2$, so that $|BU| < |BH| = \sqrt{13}/2$. Thus the diameter of $BCUFE$ does not exceed $\sqrt{13}/2$.

Therefore, the maximum diameter of the convex figure formed by the two pieces is $\sqrt{13}/2$.

(b) The diameter of the resulting figure cannot exceed the sum of the diameters of the pieces, and so is at most 2. To get a figure of diameter 2, cut the equilateral triangle into two right triangles by a median, and line them up to have their hypotenuses collinear with only one point in common.