Solutions for April

542. Solve the system of equations

$$\lfloor x \rfloor + 3\{y\} = 3.9 \ ,$$
$$\{x\} + 3\lfloor y \rfloor = 3.4 \ .$$

Solution. Let $x = a + u$ and $y = b + v$, where a and b are integers and $0 \leq u, v < 1$. Then the equations become $a + 3v = 3.9$ and $u + 3b = 3.4$. Adding the equations yields that $x + 3y = 7.3$.

Since $3b \leq 3.4 = u + 3b < 3b + 1$, it follows that $0.8 < b < 1.2$, whence $b = 1$ and $u = 0.4$. Since $a \leq 3.9 = a + 3v < a + 3$, it follows that $0.9 < a < 3.9$, whence $a = 1, 2, 3$ and $3v = 3.9 - a$. Hence $(a, v) = (1, 29/30), (2, 19/30), (3, 0.3)$.

The solutions (x, y) of the system are $(7/5, 59/30)$, $(12/5, 49/30)$ and $(17/5, 39/30) = (3.4, 1.3)$

543. Let $a > 0$ and b be real parameters, and suppose that f is a function taking the set of reals to itself for which

$$f(a^3x^3 + 3a^2bx^2 + 3ab^2x) \leq x \leq a^3f(x)^3 + 3a^2bf(x)^2 + 3ab^2f(x) \ ,$$

for all real x. Prove that f is a one-one function that takes the set of real numbers onto itself (*i.e.*, f is a *bijection*).

Solution. Let

$$g(x) = a^3x^3 + 3a^2bx^2 + 3ab^2x = (ax + b)^3 - b^3$$

for all real x. Then g is a one-one increasing function from the reals onto the reals. Let h be the composition inverse of g; then $h(g(x)) = g(h(x)) = x$ for all real x. The given condition is that

$$f(g(x)) \leq x \leq g(f(x))$$

for all real x. For each x, we have that

$$f(x) = f(g(h(x)) \leq h(x)$$

from the left inequality. Since $x \leq g(f(x))$ and h is increasing, we have that

$$h(x) \leq h(g(f(x)) = f(x) \ .$$

Therefore $f(x) = h(x)$. Applying this to the problem at hand, we find that

$$f(x) = \frac{\sqrt[3]{x + b^3} - b}{a}$$

and the result follows.

Comment. Note that it is possible to have the condition satisfied when $g(x)$ is decreasing, for example when $g(x) = -x^3$ and $f(x) = -x^{1/3}$. However, it does not seem clear that $f(x)$ is necessarily equal to $h(x)$ in this case.

544. Define the real sequences $\{a_n : n \geq 1\}$ and $\{b_n : n \geq 1\}$ by $a_1 = 1$, $a_{n+1} = 5a_n + 4$ and $5b_n = a_n + 1$ for $n \geq 1$.

(a) Determine $\{a_n\}$ as a function of n.

(b) Prove that $\{b_n : n \geq 1\}$ is a geometric progression and evaluate the sum

$$S \equiv \frac{\sqrt{b_1}}{\sqrt{b_2} - \sqrt{b_1}} + \frac{\sqrt{b_2}}{\sqrt{b_3} - \sqrt{b_2}} + \cdots + \frac{\sqrt{b_n}}{\sqrt{b_{n+1}} - \sqrt{b_n}} \ .$$

Solution. (a) We have that $a_{n+1} + 1 = 5(a_n + 1) = \cdots = 5^n(a_1 + 1) = 2 \cdot 5^n$, whence $a_n = 2 \cdot 5^{n-1} - 1$ for each positive integer n.

(b) Note that $b_n = 2 \cdot 5^{n-2}$, so that $\{b_n\}$ is a geometric progression. Since $b_{k+1} = 5b_k$, we have that $\sqrt{b_{k+1}} - \sqrt{b_k} = (\sqrt{5} - 1)\sqrt{b_k}$ for each positive integer k. it follows that $S = n/(\sqrt{5} - 1)$.

545. Suppose that x and y are real numbers for which $x^3 + 3x^2 + 4x + 5 = 0$ and $y^3 - 3y^2 + 4y - 5 = 0$. Determine $(x + y)^{2008}$.

Solution 1. The equations can be rewritten as

$$(x + 1)^3 + (x + 1) + 3 = 0$$

and

$$(y - 1)^3 + (y - 1) - 3 = 0 \ .$$

The left sides are both increasing with respect to their variables, so that x and y are uniquely determined by these equations. Adding these equations yields

$$\begin{aligned} 0 &= (x + 1)^3 + (y - 1)^3 + (x + 1) + (y - 1) \\ &= (x + y)[(x + 1)^2 - (x + 1)(y - 1) + (y - 1)^2] + (x + y) \\ &= (x + y)[x^2 - xy + y^2 + 3(x - y) + 4] \ . \end{aligned}$$

The second factor is one fourth of

$$4x^2 + 4(3 - y)x + 4(y^2 - 3y + 12) = (2x - y + 3)^2 + 3(y - 1)^2 + 4 \ ,$$

which is always positive. Hence $x + y = 0$ and the value of $(x + y)^{2008}$ is also 0.

Solution 2. As in Solution 1, we establish that each equation has exactly one real root. We note that if $x = u$ is a real solution of the first equation, then $y = -u$ is a real solution of the second. Therefore $x + y = 0$ and so $(x + y)^{2008} = 0$.

546. Let $a, a_1, a_2, \cdots, a_n$ be a set of positive real numbers for which

$$a_1 + a_2 + \cdots + a_n = a$$

and

$$\sum_{k=1}^{n} \frac{1}{a - a_k} = \frac{n + 1}{a} \ .$$

Prove that

$$\sum_{k=1}^{n} \frac{a_k}{a - a_k} = 1 \ .$$

Solution. Observe that

$$\sum_{k=1}^{n} \left(\frac{a_k}{a - a_k} - 1 \right) - n = \sum_{k=1}^{m} \frac{a}{a - a_k} - n = 1 \ .$$

from which the desired result follows.

547. Let A, B, C, D be four points on a circle, and let E be the fourth point of the parallelogram with vertices A, B, C. Let AD and BC intersect at M, AB and DC intersect at N, and EC and MN intersect at F. Prove that the quadrilateral $DENF$ is concyclic.

Solution. Since $\angle DCM = \angle BAD$, triangles DCM and ABM are similar and $AB : DC = BM : DM$. In triangle BCN, we have that

$$CN : BN = \sin \angle NBC : \sin \angle DCM \ .$$

In triangle DCM we have that

$$CM : DM = \sin \angle CDM : \sin \angle DCM \ .$$

Since

$$\angle NBC = 180^\circ - \angle ABC = \angle ADC = 180^\circ - \angle CDM \ ,$$

it follows that $CN : BN = CM : DM$.

Since $CF \| BN$, we have that $CF : BN = CM : BM$. Since $AB = EC$, we have that

$$BC \cdot CF = AB \cdot \frac{BN \cdot CM}{BM} \ .$$

Also

$$DC \cdot CN = DC \cdot \frac{CM \cdot BN}{DM} = \frac{DC}{DM} \cdot BN \cdot CM = \frac{AB}{BM} \cdot BN \cdot CM = EC \cdot CF \ .$$

From $DC \cdot CN = DC \cdot CN$, we deduce that the quadrilateral $DENF$ is concyclic.

548. In a sphere of radius R is inscribed a regular hexagonal truncated pyramid whose big base is inscribed in a great circle of the sphere (ı.e., a whose centre is the centre of the sphere). The length of the side of the big base is three times the length of the side of a small base. Find the volume of the truncated pyramid as a function of R.

Solution. The big base, being the inscribed regular hexagon in a circle of radius R, has area $(3\sqrt{3}/2)R^2$. The small base is the inscribed hexagon in a circle of radius $R/3$, and this circle is distant $\sqrt{R^2 - (R/3)^2} = 2\sqrt{2}R/3$ from the big base. We can conceive of the frustum as consisting of the full pyramid on the big base with the full pyramid on the small base taken away. The height of the former pyramid is $\sqrt{2}R$ and of the latter $\sqrt{2}R/3$. The volume of the pyramid on the big base is $(1/3)(3\sqrt{3}/2)R^2(\sqrt{2})R = (\sqrt{6}/2)R^3$. The volume of the pyramid on the small base that is removed is $1/27$ of this, so that the volume of the frustum is $(26/27)(\sqrt{6}/2)R^3 = (13\sqrt{6}/27)R^3$.

Comment. If B is the area of the big base and S the area of the small base of the frustum, then the volume of the frustum is given by $(h/3)(B + S + \sqrt{BS})$, where h is the height of the frustum. In this case $B = 9S$, so that the volume of the frustum is $(2\sqrt{2}/9)R((1/9) + 1 + (1/3))(3\sqrt{3}/2)R = (13\sqrt{6}/27)R^3$.