Solutions for January

528. Let the sequence $\{x_n : n = 0, 1, 2, \cdots\}$ be defined by $x_0 = a$ and $x_1 = b$, where a and b are real numbers, and by

$$7x_n = 5x_{n-1} + 2x_{n-2}$$

for $n \geq 2$. Derive a formula for x_n as a function of a, b and n.

Solution. This can be done by the standard theory of solving linear recursions. The auxiliary equation is $7t^2 - 5t - 2 = 0$, with roots 1 and $-2/7$. Trying a solution of the form $x_n = A \cdot 1^n + B(-2/7)^n$ and plugging in the initial conditions leads to $A + B = a$ and $A - (2/7)B = b$ and the solution

$$x_n = \frac{2a + 7b}{9} + \frac{7(a-b)}{9} \cdot \left(-\frac{2}{7}\right)^n .$$

529. Let k, n be positive integers. Define $p_{n,1} = 1$ for all n and $p_{n,k} = 0$ for $k \geq n + 1$. For $2 \leq k \leq n$, we define inductively

$$p_{n,k} = k(p_{n-1,k-1} + p_{n-1,k}) .$$

Prove, by mathematical induction, that

$$p_{n,k} = \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r (k-r)^n .$$

Solution. Let

$$q_{n,k} = \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r (k-r)^n .$$

When $n = 1$, we have that $q_{1,1} = \binom{1}{0}1 = 1$ and, for $k \geq 2$,

$$q_{1,k} = \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r k - \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r r = k[(1-1)^k - (-1)^k] + k[(1-1)^{k-1} - (-1)^{k-1}] = 0 .$$

Also $q_{n,1} = \binom{1}{0}1^n = 1$ for $n \geq 1$. When $(n, k) = (2, 2)$, we have that $q_{2,2} = \binom{2}{0}2^2 - \binom{2}{1}1 = 2$ and $p_{2,2} = 2(1 + 0) = 2$. When $n = 2$ and $k \geq 3$, then

$$\begin{aligned}
q_{2,k} &= \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r (k-r)^2 \\
&= \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r [k^2 - (2k-1)r + r(r-1)] \\
&= k^2 \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r + (2k-1)k \sum_{r=0}^{k-1} \binom{k-1}{r-1} (-1)^{r-1} + k(k-1) \sum_{r=0}^{k-1} \binom{k-2}{r-2} (-1)^{r-2} \\
&= (-1)^{k-1}k^2 + (2k-1)k(-1)^{k-2} + k(k-1)(-1)^{k-3} \\
&= (-1)^{k-3}[k^2 - 2k^2 + k + k^2 - k] = 0 .
\end{aligned}$$

Thus, we have that $p_{n,k} = q_{n,k}$ for all n and $k = 1$ as well as for $n = 1, 2$ and all k.

The remainder of the argument can be done by induction. Suppose that $n \geq 2$ and that $k \geq 2$ and that it has been shown that $p_{n,k} = q_{n,k}$ and $p_{n,k-1} = q_{n,k-1}$. Then

$$\begin{aligned}
p_{n+1,k} &= k(p_{n,k} + p_{n,k-1}) \\
&= k\Bigg[\sum_{r=0}^{k-1}\binom{k}{r}(-1)^r(k-r)^n + \sum_{r=0}^{k-2}\binom{k-1}{r}(-1)^r(k-1-r)^n\Bigg] \\
&= k\Bigg[k^n + \sum_{r=1}^{k-1}\binom{k}{r}(-1)^r(k-r)^n + \sum_{r=1}^{k-1}\binom{k-1}{r-1}(-1)^{r-1}(k-r)^n\Bigg] \\
&= k\Bigg[k^n + \sum_{r=1}^{k-1}\bigg[\binom{k}{r} - \binom{k-1}{r-1}\bigg](-1)^r(k-r)^n\Bigg] \\
&= k^{n+1} + k\sum_{r=1}^{k-1}\binom{k-1}{r}(-1)^r(k-r)^n \\
&= k^{n+1} + \sum_{r=1}^{k-1}\binom{k}{r}(-1)^r(k-r)^{n+1} \\
&= \sum_{r=0}^{k-1}\binom{k}{r}(-1)^r(k-r)^{n+1} = q_{n+1,k} \ ,
\end{aligned}$$

as desired.

530. Let $\{x_1, x_2, x_3, \cdots, x_n, \cdots\}$ be a sequence is distinct positive real numbers. Prove that this sequence is a geometric progression if and only if

$$\frac{x_1}{x_2}\sum_{k=1}^{n-1}\frac{x_n^2}{x_k x_{k+1}} = \frac{x_n^2 - x_1^2}{x_2^2 - x_1^2}$$

for all $n \geq 2$.

Solution. Necessity. Suppose that $x_k = ar^{k-1}$ for some numbers a and r. Then

$$\begin{aligned}
\frac{x_1}{x_2}\sum_{k=1}^{n-1}\frac{x_n^2}{x_k x_{k+1}} &= \frac{r^{2(n-1)}}{r}\sum_{k=1}^{n-1}\frac{1}{r^{2k-1}} \\
&= (r^{2n-3})\left(\frac{1}{r} + \frac{1}{r^3} + \cdots + \frac{1}{r^{2n-3}}\right) \\
&= 1 + r^2 + \cdots + r^{2(n-2)} = \frac{r^{2(n-1)} - 1}{r^2 - 1} = \frac{x_n^2 - x_1^2}{x_2^2 - x_1^2} \ .
\end{aligned}$$

Sufficiency. Suppose that the equations of the problem holds. When $n = 2$, both sides of the equation are equal to 1 regardless of the sequence. When $n = 3$, the equation is equivalent to

$$\frac{x_1 x_3^2}{x_1 x_2^2 x_3}(x_3 + x_1) = \frac{(x_3 - x_1)(x_3 + x_1)}{x_3^2 - x_1^2} \ .$$

Since $x_3 + x_1 \neq 0$ [why?], we can divide out this factor and multiply up the denominators to get the equivalent

$$x_3(x_2^2 - x_1^2) = x_2^2(x_3 - x_1) \Longleftrightarrow x_3 x_1^2 = x_2^2 x_1 \Longleftrightarrow x_1 x_3 = x_2^2 \ ,$$

whence x_1, x_2, x_3 are in geometric progression.

Suppose, as an induction hypothesis, for $n \geq 4$ we know that $x_k = ar^{k-1}$ for suitable a and r and $k = 1, 2, \cdots, n-1$. Let $x_n = au_n$ for some number u_n.

Then

$$\frac{u_n^2}{r}\left(\frac{1}{r} + \frac{1}{r^3} + \cdots + \frac{1}{r^{2n-5}} + \frac{1}{r^{n-2}u_n}\right) = \frac{u_n^2 - 1}{r^2 - 1}$$

$$\iff [u_n^2(1 + r^2 + r^4 + \cdots + r^{2n-6}) + r^{n-3}u_n](r^2 - 1) = (u_n^2 - 1)(r^{2n-4})$$

$$\iff (r^{2n-4} - 1)u_n^2 + (r^{n-3}r^2 - r^{n-3})u_n = (r^{2n-4})u_n^2 - r^{2n-4}$$

$$\iff 0 = u_n^2 - (r^{n-1} - r^{n-3})u_n - r^{2n-4} = (u_n - r^{n-1})(u_n + r^{n-3}) \ .$$

The case $u_n = -r^{n-3}$ is rejected because of the condition that the sequence consists of positive terms. Hence $u_n = r^{n-1}$, as desired. The result follows.

Comment. In the absence of the positivity contiion, the second root of the quadratic can be used. For example, the finite sequences $\{1, r, r^2, -r, 1\}$ and $\{1, r, r^2, -r, -r^2\}$ both satisfies the equations for $2 \leq n \leq 5$. It would be interesting to investigate the situation further.

531. Show that the remainder of the polynomial

$$p(x) = x^{2007} + 2x^{2006} + 3x^{2005} + 4x^{2004} + \cdots + 2005x^3 + 2006x^2 + 2007x + 2008$$

is the same upon division by $x(x+1)$ as upon division by $x(x+1)^2$.

Solution 1. We have that

$$\begin{aligned} p(x) &= (x^{2007} + 2x^{2006} + x^{2005}) + 2(x^{2005} + 2x^{2004} + x^{2003}) + 3(x^{2003} + \\ &\quad 2x^{2002} + x^{2001}) + \cdots + 1003(x^3 + 2x^2 + x) + 1004x + 2008 \\ &= x(x+1)^2(x^{2004} + 2x^{2002} + 3x^{2000} + \cdots + 1003) + (1004x + 2008) \ , \end{aligned}$$

from which the result follows with remainder $1004x + 2008$.

532. The angle bisectors BD and CE of triangle ABC meet AC and AB at D and E respectively and meet at I. If $[ABD] = [ACE]$, prove that $AI \perp ED$. Is the converse true?

Solution. Observe that

$$[ADB] : [CBD] = AD : DC = AB : BC$$

and that

$$[ACE] : [BCE] = AE : EB = AC : BC \ .$$

Now

$$\begin{aligned} [ABD] = [ACE] &\iff [DBC] = [ABC] - [ABD] = [ABC] - [ACE] = [EBC] \\ &\iff ED \| BC \iff AE : EB = AD : DC \\ &\iff AB : BC = AC : BC \iff AB = BC \\ &\iff AI \perp BC \ . \end{aligned}$$

Both the result and the converse is true. If $[ABD] = [ACE]$, the foregoing chain of implications can be read in the forward direction to deduce that $AI \perp ED$. Note that AI bisects angle A in triangle AED. Thus, if $AI \perp ED$, then it follows that triangle AED is isosceles with $AE = AD$. Then $AE : DC = AD : DC = AB : BC$ and $AE : EB = AC : BC$, whence $DC \cdot AB = AE \cdot BC = EB \cdot AC$. Therefore $DC \cdot (AE + EB) = EB \cdot (AD + CD)$, so that $DC \cdot AE = EB \cdot AD$ and $DC = EB$. Therefore $AB = AC$ and, following the foregoing implication in the backwards direction, we find that $[ABD] = [ACE]$.

533. Prove that the number

$$1 + \lfloor (5 + \sqrt{17}))^{2008} \rfloor$$

is divisible by 2^{2008}.

Solution. Let $a = 5 + \sqrt{17}$ and $b = 5 - \sqrt{17}$, so that $a + b = 10$ and $ab = 8$. Define $x_n = a^n + b^n$. Then $x_1 = 10$, $x_2 = (a + b)^2 - 2ab = 96$ and

$$\begin{aligned} x_{n+2} &= a^{n+2} + b^{n+2} = (a + b)(a^{n+1} + b^{n+1}) - ab(a^n + b^n) \\ &= 10x_{n+1} - 8x_n \ , \end{aligned}$$

for $n \geq 0$. Note that x_1 is divisible by 2 and x_2 by 4. Suppose, as an induction hypothesis, that $x_n = 2^n u$ and $x_{n+1} = 2^{n+1}v$, for some $k \geq 0$ and integers u and v. Then

$$x_{n+2} = 5 \cdot 2^{n+2} - 2^{n+3} = 3 \cdot 2^{n+2} \ .$$

Hence, for all positive integers n, 2^n divides x_n.

Observe that $(5 - \sqrt{17})^n = b^n < 1$ for each positive integer n and that $a^n + b^n$ is a positive integer. Therefore $x_n = a^n + b^n > a^n > a^n + b^n - 1 = x_n - 1$, whence $x_n = 1 + \lfloor a^n \rfloor$ and the result follows.

534. Let $\{x_n : n = 1, 2, \cdots\}$ be a sequence of distinct positive integers, with $x_1 = a$. Suppose that

$$2\sum_{k=1}^{n} \sqrt{x_i} = (n + 1)\sqrt{x_n}$$

for $n \geq 2$. Determine $\sum_{k=1}^{n} x_k$.

Solution. When $n = 2$, $2(\sqrt{x_1} + \sqrt{x_2}) = 3\sqrt{x_2}$, whence $\sqrt{x_2} = 2\sqrt{x_1}$ and $x^2 = 4x_1 = 4a$. When $n = 3$,

$$2(\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}) = 4\sqrt{x_3} \Longrightarrow 2\sqrt{x_3} = 2(\sqrt{x_1} + \sqrt{x_2}) = 6\sqrt{x_1} \Longrightarrow x_3 = 9x_1 = 9a \ .$$

We conjecture that $x_k = k^2 a$ for each positive integer k.

Let $m \geq 2$ and suppose that $x_k = k^2 a$ for $1 \leq k \leq m - 1$.

$$\begin{aligned} (m - 1)\sqrt{x_m} &= (m + 1)\sqrt{x_m} - 2\sqrt{x_m} = 2(\sqrt{x_1} + \cdots + \sqrt{x_{m-1}}) \\ &= 2\sqrt{x_1}(1 + 2 + \cdots + (m - 1)) = m(m - 1)\sqrt{a} \ , \end{aligned}$$

whence $\sqrt{x_m} = m\sqrt{a}$ and $x_m = m^2 a$.

Thus, $x_k = k^2 a$ for all $k \geq 1$. Therefore

$$\sum_{k=1}^{n} x_k = (1 + 4 + \cdots + n^2)a = \frac{n(n + 1)(2n + 1)a}{6} \ .$$