Solutions for September

563. (a) Determine infinitely many triples (a,b,c) of integers for which a,b,c are not in arithmetic progression and $ab+1$, $bc+1$, $ca+1$ are all squares.

(b) Determine infinitely many triples (a,b,c) of integers for which a,b,c are in arithemetic progression and $ab+1$, $bc+1$, $ca+1$ are all squares.

(c) Determine infinitely many triples (u,v,w) of integers for which $uv-1$, $vw-1$, $wu-1$ are all squares. (Can it be arranged that u,v,w are in arithmetic progression?)

Solution. (a) Here are some families of solutions that are (mostly) not in arithmetic progression, where n is an integer:

$$(0,0,n); (0,n-1,n+1); (0,2,2n(n+1)); (1,n^2-1,n^2+2n); (n-1,n+1,4n); (n,n+2,4(n+1));$$

$$(m, mn^2+2n, m(n+1)^2+2(n+1)); (f_{2(n-1)}, f_{2n}, f_{2(n+1)}) \ .$$

Here, $\{f_n\}$ is the Fibonacci sequence defined by $f_1=f_2=1$ and $f_{n+1}=f_n+f_{n-1}$ for every integer n. We need to establish that $f_{2n}f_{2n+2}+1=f_{2n+1}^2$ and $f_{2n-2}f_{2n+2}+1=f_{2n}^2$ for each integer n. Since

$$f_{2n+1}^2-f_{2n}^2=f_{2n+2}f_{2n-1}=f_{2n+2}(f_{2n}-f_{2n-2})=(f_{2n}f_{2n+2}+1)-(f_{2n-2}f_{2n+2}+1) \ ,$$

the two equations are equivalent. Note that

$$f_{2n}f_{2n+2}-f_{2n+1}^2=f_{2n}(f_{2n+1}+f_{2n})-f_{2n+1}^2=f_{2n+1}(f_{2n}-f_{2n+1})+f_{2n}^2=-(f_{2n+1}f_{2n-1}-f_n^2) \ ;$$

a proof by induction can be devised for the first equation.

(b) (i) Some examples for (a,b,c) are $(-1,0,1)$, $(0,2,4)$, $(1,8,15)$, $(4,30,56))$, $(15,112,209)$. This suggests the possibility $(u_n, 2u_{n+1}, u_{n+2})$ where $u_0=0$, $u_1=1$, $u_2=4$ and $u_{n+1}=4u_{n-1}-u_n$ for integral n. Since $u_{n+1}-2u_n=2u_n-u_{n-1}$, $u_{n-1}, 2u_n, u_{n+1}$ are in arithmetic progression.

We now prove, for each integer n,

$$2u_nu_{n+1}+1=(u_{n+1}-u_n)^2 \tag{1}$$

$$u_{n+2}u_n+1=u_{n+1}^2 \tag{2}$$

$$2u_{n+1}u_{n+2}+1=(u_{n+2}-u_{n+1})^2 \tag{3}$$

Properties (1) and (3) are the same. The truth of (2) is equivalent to the truth of (1), since

$$\begin{aligned}[(2u_nu_{n+1}+1)-(u_{n+1}-u_n)^2)]&+[(u_nu_{n+2}+1)-u_{n+1}^2] \\ &=u_n(2u_{n+1}-u_{n+2})+u_n(2u_{n+1}-u_n) \\ &=-u_n(u_{n+2}-4u_{n+1}+u_n)=0 \ .\end{aligned}$$

We establish (2) by induction. Since

$$\begin{aligned}u_{n+2}u_n+1-u_{n+1}^2&=u_n(4u_{n+1}-u_n)+1-u_{n+1}^2 \\ &=u_{n+1}(4u_n-u_{n+1})+1-u_n^2 \\ &=u_{n+1}u_{n-1}+1-u_n^2 \ ,\end{aligned}$$

$u_{n+2}u_n+1-u_{n+1}^2=u_2u_0+1-u_1^2=0$ for all n. The desired results follow.

(b) (ii) [A. Dhawan] Let $v^2 - 3u^2 = 1$ for some integers v and u. Then, if $(a, b, c) = (2u - v, 2u, 2u + v)$, then

$$\begin{aligned} ab + 1 &= (2u - v)2u + 1 = 4u^2 - 2uv + 1 \\ &= u^2 + (v^2 - 1) - 2uv + 1 = (u - v)^2 \ ; \end{aligned}$$

$$\begin{aligned} bc + 1 &= 2u(2u + v) + 1 = 4u^2 + 2uv + 1 \\ &= u^2 + 2uv + v^2 - 1 + 1 = (u + v)^2 \ ; \end{aligned}$$

and

$$\begin{aligned} ac + 1 &= (2u - v)(2u + v) + 1 \\ &= 4u^2 - v^2 + 1 = u^2 \ . \end{aligned}$$

(Note that in this solution, the roots of the square, not all positive, are also in arithmetic progression.)

The equation $v^2 - 3u^2 = 1$ is a Pell's equation with infinitely many solutions given by $(v, u) = (x_n, y_n)$, where $x_n + y_n\sqrt{3} = (2 + \sqrt{3})^n$, for positive integers n.

(b) (iii) We look for solutions in which one integer is 0, Thus (a, b, c) has the form $(0, p, 2p)$, where $2p^2 + 1 = q^2$. This is a Pell's equation whose solutions are given by $(q, p) = (x_n, y_n)$ where $x_n + y_n\sqrt{2} = (3 + 2\sqrt{2})^n$ for positive integers n. P. Wen also identified triples $(0, p, 2p)$ where $2p^2 + 1$ is square. Since this square is odd, it must have the form $(2y + 1)^2$, so that $p^2 = 2y(y + 1)$. Thus, p is even, say $p = 2x$, and so $x^2 = \frac{1}{2}y(y + 1)$, which is at once a square and a triangula number. Conversely, any triangular number which is also a square gives a solution triple, so we need to know that there are infinitely many such. If $x^2 = \frac{1}{2}y(y + 1)$, then $8x^2 + 1 = (2y + 1)^2$, so that

$$\begin{aligned} [x(2y + 1)]^2 = 4p^2(2y + 1)^2 &= \frac{4y(y + 1)(2y + 1)^2}{2} \\ &= \frac{1}{2}(4y^2 + 4y)(4y^2 + 4y + 1) \ . \end{aligned}$$

Starting with $(x, y) = (1, 1)$, we are led to $(6, 8), (204, 288)$ and so on, and obtain the solutions $(a, b, c) = (0, 2, 4), (0, 12, 24), (0, 408, 816), \cdots$.

(c) Here are some families of solutions for (u, v, w): $(1, 1, n^2 + 1)$, $(1, n^2 + 1, (n + 1)^2 + 1)$ along with $(f_{2n-1}, f_{2n+1}, f_{2n+3})$, where f_n is the Fibonacci sequence defined in the solution to (a). S.H. Lee found a two-parameter family:

$$(m^2 + 1, (m^2 + 1)n^2 + 2mn + 1, (m^2 + 1)(n + 1)^2 + 2m(n + 1) + 1) \ .$$

In this case,

$$uv - 1 = [(m^2 + 1)n + m^2]^2 \ ; \qquad uw - 1 = [(m^2 + 1)(n + 1) + m]^2 \ ;$$

$$uw - 1 = [(m^2 + 1)n(n + 1) + 2mn + m + 1]^2 \ .$$

J. Zung identified the triple $(1, 2, p^2 + 1)$ where $q^2 - 2p^2 = 1$ for some integer q. [Exercise: Check these out.]

564. Let $x_1 = 2$ and

$$x_{n+1} = \frac{2x_n}{3} + \frac{1}{3x_n}$$

for $n \geq 1$. Prove that, for all $n > 1$, $1 < x_n < 2$.

Solution 1. Since, for $n \geq 1$,

$$x_{n+1} - 1 = \frac{2x_n}{3} + \frac{1}{3x_n} - 1 = \frac{2x_n^2 - 3x_n + 1}{3x_n} = \frac{(2x_n - 1)(x_n - 1)}{3x_n} \ ,$$

it can be shown by induction that $x_n > 1$ for all $n \geq 1$.

Since, for $n \geq 1$,

$$x_n - x_{n+1} = \frac{x_n^2 - 1}{3x_n} ,$$

it follows that $x_{n+1} < x_n \leq 2$ for all $n \geq 1$. Thus we have the desired inequality.

Solution 2. Observe that $x_2 = 3/2$. As an induction hypothesis, suppose that $1 < x_n < 2$ for some $n \geq 2$. Then

$$x_{n+1} = \frac{x_n}{3} + \frac{1}{3}\left(x_n + \frac{1}{x_n}\right) > \frac{1}{3} + \frac{1}{3} \cdot 2 = 1$$

by the arithmetic-geometric means inequality. Also,

$$x_{n+1} = \frac{2x_n}{3} + \frac{1}{3x_n} < \frac{4}{3} + \frac{1}{3} = \frac{5}{3} < 2 .$$

The result follows.

Comment. An induction argument for the right inequalitycan be based on the observation that, when $1 < x < 2$, the quadratic $2x^2 - 6x + 1 = 2(x - \frac{3}{2})^2 - \frac{7}{2} \leq \frac{1}{2} - \frac{7}{2} = -3 < 0$, whence $2x^2 + 1 < 6x$ and $(2x_n/3) + (1/3x^n) < 2$.

Solution 3. Let $f(x) = (2x/3) + (1/3x)$. Then $f'(x) = (2/3) - (1/3x^3) \geq 1/3 > 0$ when $1 \leq x$. Hence $f(x)$ is strictly increasing on the interval $[1, 2]$ and so takes values strictly between $f(1) = 1$ and $f(2) = 3/2$ on the open interval $(0, 1)$. Since $x_2 \in (0, 1)$ and $x_{n+1} = f(x_n)$ for $n \geq 1$, the desired result can be established by induction.

565. Let ABC be an acute-angled triangle. Points A_1 and A_2 are located on side BC so that the four points are ordered B, A_1, A_2, C; similarly B_1 and B_2 are on CA in the order C, B_1, B_2, A and C_1 and C_2 on side AB in order A, C_1, C_2, B. All the angles AA_1A_2, AA_2A_1, BB_1B_2, BB_2B_1, CC_1C_2, CC_2C_1 are equal to θ. Let $\mathfrak{T}_1$ be the triangle bounded by the lines AA_1, BB_1, CC_1 and $\mathfrak{T}_2$ the triangle bounded by the lines AA_2, BB_2, CC_2. Prove that all six vertices of the triangles are concyclic.

Solution 1. Let $A_0B_0C_0$ be the triangle with $B_0C_0 \| BC$, $C_0A_0 \| CA$, $A_0B_0 \| AB$ where A, B, C are the respective midpoints of B_0C_0, C_0A_0, A_0B_0. Then the orthocentre H of triangle ABC is the circumcentre of triangle $A_0B_0C_0$.

Suppose that K is the intersection point of AA_2 and BB_2. Since the exterior angle at A_2 is equal to the interior angle at B_2, the quadrilateral A_2KB_2C is concyclic, so that $\angle BKA_2 = \angle BCA = \angle BC_0A$. Therefore, the quadrilateral AC_0BK is concyclic; the quadrilateral AC_0BH with right angles at A and B is concyclic. Thus, BC_0AKH is concyclic and so $\angle C_0KH = \angle C_0AH = 90°$.

Since $C_0A_0 \| AC$, $\angle C_0HK = \angle C_0BK = \angle BB_2C = \theta$. Therefore $|HK| = |HC_0| \cos\theta = R\cos\theta$, where R is the circumradius of triangle $A_0B_0C_0$. The same argument can be applied to the intersection point of any pairs (AA_i, BB_i), (BB_i, CC_i), (CC_i, AA_i) $(i = 1, 2)$. All the vertices lie on the circle with centre H and radius R.

Solution 2. [A. Murali] Let $AA_1 \cap BB_1 = P$, $BB_1 \cap CC_1 = Q$, $CC_1 \cap AA_1 = R$, $AA_2 \cap BB_2 = V$, $BB_2 \cap CC_2 = W$, $CC_2 \cap AA_2 = U$. We have that

$$\begin{aligned}\angle A_2CU &= \angle BCC_2 = 180° - \angle ABC - \angle BC_2C \\ &= 180° - \angle ABC - (180° - \theta) = \theta - \angle ABC ,\end{aligned}$$

and

$$\begin{aligned}\angle CUA_2 &= 180° - (\angle A_2CU + \angle AA_2C) \\ &= 180° - (\theta - \angle ABC) - (180° - \theta) = \angle ABC .\end{aligned}$$

Since $\angle AA_1B = \angle CA_2U$ and $\angle ABA_1 = \angle ABC = \angle A_2UC$, triangles AA_1B and CA_2U are similar. Therefore $CA_2 : A_2U = AA_1 : BA_1$, from which

$$|A_2U| = \frac{|CA_2| \times |BA_1|}{|AA_1|} \ .$$

Similarly, $\angle BPA_1 = \angle BCA$, which along with $\angle BA_1A = \angle BB_1C$ implies that triangles BA_1P and BB_1C are similar. Therefore

$$|PA_1| = \frac{|BA_1| \times |B_1C|}{|BB_1|} \ .$$

Hence,

$$\frac{|A_2U|}{|A_1P|} = \frac{|CA_2| \times |BB_1|}{|AA_1| \times |B_1C|} = \frac{|CA_2| \times |BB_1|}{|AA_2| \times |B_1C|} \ .$$

Since triangles CBB_1 and CAA_2 are similar, $CA_2 : AA_2 = CB_1 : BB_1$, from which it follows that $UA_2 = PA_1$, so that $UA_2 : AA_2 = PA_1 : AA_1$ and $PU \| A_1A_2$.

Similarly, $QV \| B_1B_2$. Therefore

$$\angle PUV = \angle A_1A_2A = \theta = \angle B_2B_1B = \angle VQP$$

and $PUQV$ is concyclic.

Since $\angle C_2UA = \angle CUA_2 = \angle ABC$ and $\angle AC_2U = \theta = \angle BC_1C$, triangles AUC_2 and CBC_1 are similar, so that

$$|UC_2| = \frac{|BC_1| \times |C_2A|}{|C_1C|} \ .$$

Since triangles BQC_1 and CAC_2 are similar,

$$|QC_1| = \frac{|BC_1| \times |AC_2|}{|CC_2|} \ .$$

Since $CC_2 = CC_1$, $UC_2 = QC_1$ so that $UQ \| BA$. Similarly $WP \| AC$. Therefore, $\angle WPQ = 180° - \theta = \angle WUQ$, and $WPUQ$ is concyclic.

Since $RWPU$, $WPUQ$ and $PUQV$ are all concyclic, R and Q lie on the circle through W, P, U and W and V lie on the circle through P, U, Q. The result follows.

Solution 3. [P. Wen] Use the notation of Solution 2 and let H denote the orthocentre of triangle ABC. Since $\angle WBH = 90° - \theta = \angle WCH$, the points B, W, H, C are concyclic; similarly, B, H, Q, C are concyclic. Hence B, W, H, Q, C are concyclic. Similarly, A, V, H, P, B are concyclic.

Since

$$\begin{aligned} \angle PQW &= \angle BQW = \angle BCW = \angle BCC_2 = \angle BAA_1 \\ &= \angle BAP = \angle BVP = \angle PVW \ , \end{aligned}$$

the points P, Q, V, W are concyclic. Since $\angle BHW = \angle BCW = \angle BAP = \angle BHP$, $\angle HBW = \angle HBB_2 = \angle HBB_1 = \angle HBP$, and side BH is common, the triangles BHW and BHP are congruent, so that $BP = BW$.

Since $\angle PHW = 2\angle BHW = 2\angle PQW$, H must be the centre of the circle through P, Q, V, W, so that H is equidistant from these four points. Similarly, H is equidistant from the four points R, P, U, V and from the points Q, R, W, U. The desired result follows.

Solution 4. [P.J. Zhao] Use the notation of Solutions 2 and 3, with H the orthocentre of triangle ABC. Since the quadrilaterals BC_1RA_1, BC_1B_2C and CA_2VB_2 are concyclic, we have that

$$AR : AA_1 = AC_1\text{"}AB = AB_2 : AC = AV : AA_2 \ .$$

Since $AA_1 = AA_2$, $AR = AV$. As AA_1A_2 is isosceles, AH bisects angle A_1AA_2 and triangles AHR amd AHV are congruent (SAS), so that $HR = HV$. Similarly, $HP = HW$ and $HQ = HU$.

Since the quadrilaterals CB_1PA_1, BC_2B_1C and BC_2UA_2 are concyclic, it follows that

$$AP : AA_1 = AB_1 : AC = AC_2 : AB = AU : AA_2 \ ,$$

whence $AP = AU$. Since triangles AHP and AHU are congruent, $HP = HU$. Similarly, $HQ = HV$ and $HR = HW$.

Thus, all six vertices of the two triangles are equidistant from H and the result follows.

Comment. J. Zung observed that a rotation about H through the angle 2θ takes the line AA_1 onto the line A_2A, the line BB_1 onto the line B_2B and the line CC_1 onto the line C_2C. To see this, note that if A_3 and A_4 are the feet of the perpendiculars dropped from H to AA_1 and AA_2 respectively, then

$$\angle A_3HA_4 = \angle A_3HA + \angle AHA_4 = \angle AA_1A_2 + \angle AA_2A_1 = 2\theta \ .$$

This rotation takes $P \rightarrow V$, $Q \rightarrow W$, $R \rightarrow U$, so that $HP = HV$, $HQ = HW$, $HR = HU$. This taken with either half of the argument of Solution 4 yields the result.

566. A deck of cards numbered 1 to n (one card for each number) is arranged in random order and placed on the table. If the card numbered k is on top, remove the kth card counted from the top and place it on top of the pile, not otherwise disturbing the order of the cards. Repeat the process. Prove that the card numbered 1 will eventually come to the top, and determine the maximum number of moves that is required to achieve this.

Solution For each card, a move must result in exactly one of the following possibilities: (i) the card remains in the same position; (ii) the card moves one position lower in the deck; (iii) the card is brought to the top of the deck.

We prove by induction the following statement: *Suppose that we have deck of m cards each with a different number, and that we follow the procedure of the problem; then after at most $2^{m-1} - 1$ moves the process will have to stop either because card 1 comes to the top or a card with a number exceeding m comes to the top.* It is straightfoward to see that the result holds for $m = 1$ and $m = 2$. Suppose that when $1 \leq m \leq r - 1$.

Let $m = r$. Since there are r cards with different numbers, there is a card u where either $u = 1$ or $u > r$. Suppose that u occurs in the kth position. Then the first $k - 1$ positions must contain card 1 or a card exceeding $k - 1$. By the induction hypothesis, in at most $2^{k-2} - 1$ moves one of the following must occur: (1) the process stops because a card numbered 1 or with a number exceeding m (possibly u) comes to the top, or (2) a card with a number between $k + 1$ and m inclusive comes to the top. In the second case, one more move will cause u to go to the $(k+1)$th position. Therefore, after at most $1+2+\cdots+2^{r-3} = 2^{r-2}-1$, either the process has stopped or u has been forced from the $(r - 1)$th position to the rth position.

The top $r-1$ cards must contain at least one lying outside of the range $[2, r-1]$. Therefore, in at most $2^{r-2} - 1$ further moves, either the process stops, because card number 1 or a card with a number exceeding r comes to the top, or else r comes to the top. In the latter case, one further move will make u come to the top. Thus, we can get a card with either the number 1 or a card exceeding m to the top in at most $(2^{r-2} - 1) + (2^{r-2} - 1) + 1 = 2^{r-1} - 1$ moves.

The desired result is a special case of this, where $m = n$ and the card outside of the range $[2, n]$ is the card numbered 1.

There is an initial arrangement of the cards where the maximum number of moves is attained, namely $(n, 1, n-1, n-2, \cdots, 3, 2)$. To show this, we establish the following result:

Let $m \geq 2$. Then the sequence $(m, u, m-1, m-2, \cdots, 2)$ becomes the sequence $(u, m, m-1, m-2, \cdots, 2)$ in exactly $2^{m-1} - 1$ moves, where u is any number.

This is true for $m = 2$ ($(2, u) \rightarrow (u, 2)$) and $m = 3$ ($(3, u, 2) \rightarrow (2, 3, u) \rightarrow (3, 2, u) \rightarrow (u, 3, 2)$). Assume that $m \geq 4$ and that the result holds for all values of m up to and including $k - 1$. Then we can use the induction hypothesis to make changes as follows (where the number in square brackets indicates the number of moves):

$$\begin{aligned}
(k, u, k-1, \cdots, 2) &\rightarrow [1] \rightarrow (2, k, u, k-1, \cdots 3) \rightarrow [1] \rightarrow (k, 2, u, k-1, \cdots 3) \\
&\rightarrow [1] \rightarrow (3, k, 2, u, k-1, \cdots, 4) \rightarrow [3] \rightarrow (k, 3, 2, u, k-1, \cdots, 4) \\
&\rightarrow [1] \rightarrow (4, k, 3, 2, u, k-1, \cdots, 5) \rightarrow [7] \rightarrow (k, 4, 3, 2, u, k-1, \cdots, 5) \\
&\vdots \\
&\rightarrow [1] \rightarrow (j, k, j-1, \cdots, 2, u, k-1, \cdots, j+1) \\
&\rightarrow [2^{j-1} - 1] \rightarrow (k, j, j-1, \cdots, 2, u, k-1, \cdots, j+1) \\
&\vdots \\
&\rightarrow [1] \rightarrow (k-1, k, k-2, \cdots, 3, 2, u)) \rightarrow [2^{k-2} - 1] \rightarrow (k, k-1, k-2, \cdots, 3, 2, u) \\
&\rightarrow [1] \rightarrow (u, k, k-1, \cdots, 3, 2) \ .
\end{aligned}$$

The total number of moves is

$$1 + \sum_{j=2}^{k-2} [(2^{j-1} - 1) + 1] = 1 + 2 + \cdots + 2^{k-2} = 2^{k-1} - 1 \ .$$

In particular, when $u = 1$ and $k = n$, we conclude that $(n, 1, n-1, \cdots, 2)$ goes to $(1, n, n-1, \cdots, 2)$ in $2^{n-1} - 1$ moves.

Comment. A. Abdi provided the following induction argument that the process must terminate. The result clearly holds for $n = 1$. Suppose it holds for $1 \leq n \leq m - 1$, If card 1 never comes to the top, then the process never terminates and card 1 eventually finds its way to position $r \leq m$ and stays there. The cards below position r (if any) never move from that point on. Let X be the set of cards on top of 1 at that point whose numbers exceed r and Y the set of cards on top of 1 whose numbers do not exceed r, so that $\#X + \#Y = r - 1$. Since card 1 cannot move down, the cards in X never come to the top, so it is immaterial what numbers appear on these cards. Relabel these cards with numbers from $\{2, 3, 4, \cdots, r\}$ that do not belong to the cards in Y, so that the numbers from 2 to r inclusive all appear on top of card 1. These cards get permuted among themselves by subsequent moves.

However, by the induction hypothesis applied to this deck of $r - 1 \leq m - 1$ cards atop card 1 (with card r relabelled to a second card 1), we see that card r must eventually come to the top, when then will force card 1 to come to the top. This yields a contradiction of the assertion that the process can go on forever.

567. (a) Let A, B, C, D be four distinct points in a straight line. For any points X, Y on the line, let XY denote the *directed* distance between them. In other words, a positive direction is selected on the line and $XY = \pm|XY|$ according as the direction X to Y is positive or negative. Define

$$(AC, BD) = \frac{AB/BC}{AD/DC} = \frac{AB \times CD}{BC \times DA} \ .$$

Prove that $(AB, CD) + (AC, BD) = 1$.

(b) In the situation of (a), suppose in addition that $(AC, BD) = -1$. Prove that

$$\frac{1}{AC} = \frac{1}{2}\left(\frac{1}{AB} + \frac{1}{AD}\right) \ ,$$

and that

$$OC^2 = OB \times OD \ ,$$

where O is the midpoint of AC. Deduce from the latter that, if Q is the midpoint of BD and if the circles on diameters AC and BD intersect at P, $\angle OPQ = 90^\circ$.

(c) Suppose that A, B, C, D are four distinct on one line and that P, Q, R, S are four distinct points on a second line. Suppose that AP, BQ, CR and DS all intersect in a common point V. Prove that $(AC, BD) = (PR, QS)$.

(d) Suppose that $ABQP$ is a quadrilateral in the plane with no two sides parallel. Let AQ and BP intersect in U, and let AP and BQ intersect in V. Suppose that VU and PQ produced meet AB at C and D respectively, and that VU meets PQ at W. Prove that

$$(AB, CD) = (PQ, WD) = -1 \ .$$

Solution. (a)

$$\begin{aligned}\frac{AC \times BD}{CB \times DA} + \frac{AB \times CD}{BC \times DA} &= \frac{(AB + BC) \times (BC + CD) - AB \times CD}{BC \times AD} \\ &= \frac{BC \times (AB + BC + CD)}{BC \times AD} = 1 \ .\end{aligned}$$

(b) $AB \times CD = BC \times AD \Longrightarrow$

$$AB \times (AD + CA) = (BA + AC) \times AD \Longrightarrow 2AB \times AD = AB \times AC + AC \times AD$$

$$\Longrightarrow \frac{1}{AC} = \frac{1}{2}\left(\frac{1}{AB} + \frac{1}{AD}\right) .$$

Since $AB = AO + OB = OC + OB$, $AD = AO + OD = OC + OD$ and $AC = 2OC$,

$$\frac{1}{OC} = \frac{1}{OB + OC} + \frac{1}{OD + OC} \ ,$$

from which the desired result follows. Since $OP = OC^2$, $OP^2 = OB \times OD$, so that OP is tangent to the circle of diameter BD. Hence $PQ \perp OP$ and the result follows.

Comment. For the last part, M. Sardarli noted that

$$\begin{aligned}OP^2 + PQ^2 &= OC^2 + BQ^2 = OB \times OD + BQ^2 = (OQ + QB)(OQ - QB) + BQ^2 \\ &= OQ^2 - QB^2 + BQ^2 = OQ^2 \ ,\end{aligned}$$

whence $\angle OPQ = 90^\circ$.

(c) First observe that, of both lines lie on the same side of V, then corresponding lengths among A, B, C, D and P, Q, R, S have the same signs, while if V is between the lines, then the signs are opposite. Let a, b, c, d be the respective lengths of AV, BV, CV, DV; let $\alpha, \beta, \gamma, \delta$ be the respective angles AVB, CVD, BVC, DVA; let h be the distance from V to the line $ABCD$. Then

$$\begin{aligned}|(AC, BD)| &= \left|\frac{AB \times CD}{BC \times DA}\right| = \left|\frac{(\frac{1}{2}h \times AB) \times (\frac{1}{2}h \times CD)}{(\frac{1}{2}h \times BC) \times (\frac{1}{2}h \times DA)}\right| \\ &= \frac{[AVB] \times [CVD]}{[BTC] \times [DTA]} = \frac{(\frac{1}{2}ab \sin \alpha)(\frac{1}{2}cd \sin \beta)}{(\frac{1}{2}bc \sin \gamma)(\frac{1}{2}ad \sin \delta)} \\ &= \frac{\sin \alpha \sin \beta}{\sin \gamma \sin \delta} \ .\end{aligned}$$

Since $\angle AVB = \angle PVQ$, *etc.*, we find that $|(PR, QS)| = (\sin\alpha\sin\beta)/(\sin\gamma\sin\delta)$, and the result follows.

(d) By (c), with the role of V played respectively by V and U, we obtain that

$$(AB, CD) = (PQ, WD) = (BA, CD) = \frac{1}{(AB, CD)} ,$$

so that $(AB, CD)^2 = 1$. Since $(AB, CD) + (AC, BD) = 1$ and (AC, BD) can vanish only if $A = B$ or $C = D$, we must have that $(AB, CD) = -1$.

568. Let ABC be a triangle and the point D on BC be the foot of the altitude AD from A. Suppose that H lies on the segment AD and that BH and CH intersect AC and AB at E and F respectively.

Prove that $\angle FDH = \angle HDE$.

Solution 1. Suppose that $ED \| AB$. Then by Ceva's theorem,

$$1 = \frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = \frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CD|}{|DB|} ,$$

so that $AF = FB$. Hence F is the circumcentre of the right triangle ADB, so that $AF = DF$ and $\angle FDB = \angle FAD = \angle HDE$.

Otherwise, let AB and ED produced intersect at G. Then, in the notation of problem 567, $(AB, FG) = -1$. Therefore

$$\sin\angle ADF \sin\angle BDG = \sin\angle FDB \sin\angle ADG = \cos\angle ADF \cos\angle BDG .$$

Hence $\tan\angle ADF = \cot\angle BDG = \tan\angle ADE$ and $\angle ADF = \angle ADE$.

Solution 2. Suppose that DE and DF intersect the line through A parallel to BC at the points M and N respectively. Since triangles BDF and ANF are similar, as are triangles CDE and AME,

$$\frac{|AM|}{|AN|} = \frac{|AM|}{|CD|} \cdot \frac{|CD|}{|DB|} \cdot \frac{|DB|}{|AN|} = \frac{|AE|}{|EC|} \cdot \frac{|CD|}{|DB|} \cdot \frac{|BF|}{|AF|} = 1 ,$$

by Ceva's theorem. Therefore, $AM = AN$, so that triangles AMD and AND are congruent and $\angle ADF = \angle ADM$.

Solution 3. [R. Peng] Suppose that the points K and L are selected on BC so that $EK \perp CB$ and $FL \perp BC$. Let $X = CH \cap EK$ and $Y = BH \cap FL$. Then $FL \| EK$, so that the triangles FYH and XEH with respective heights LD and KD are similar. Therefore

$$\begin{aligned} LD : DK &= FY : EX = (FY : AH)(AH : EX) = (BF : BA)(CA : EC) \\ &= (FL : AD)(AD : EK) = FL : EK . \end{aligned}$$

Therefore triangles FLD and EKD are similar, so that $\angle LDF = \angle KDE$. The result follows.

Solution 4. [A. Murali] Suppose that $\angle FDH = \alpha$ and $\angle HDE = \beta$. By the Law of Sines,

$$\frac{|AF|}{\sin\alpha} = \frac{|AD|}{\sin\angle AFD}$$

and

$$\frac{|AE|}{\sin\beta} = \frac{|AD|}{\sin\angle AED} .$$

Therefore
$$\frac{\sin\alpha}{\sin\beta} = \frac{|AF|}{|AE|}\cdot\frac{\sin\angle AFD}{\sin\angle AED} = \frac{|AF|}{|AE|}\cdot\frac{\sin\angle BFD}{\sin\angle CED} \ .$$

Since
$$\frac{|BD|}{\sin\angle BFD} = \frac{|BF|}{\sin\angle BDF} = \frac{|BF|}{\cos\alpha}$$
and
$$\frac{|CD|}{\sin\angle CED} = \frac{|CE|}{\cos\beta} \ ,$$
it follows, using Ceva's theorem, that
$$\frac{\sin\alpha}{\sin\beta} = \frac{|AF|}{|AE|}\cdot\frac{|BD|}{|BF|}\cdot\frac{|CE|}{|CD|}\cdot\frac{\cos\alpha}{\cos\beta} = \frac{|AF|}{|BF|}\cdot\frac{|BD|}{|CD|}\cdot\frac{|CE|}{|AE|}\cdot\frac{\cos\alpha}{\cos\beta} \ .$$
Therefore $\tan\alpha = \tan\beta$ and the desired result follows.

Comment. It was intended that D be an interior point of BC. However, in the case that either B or C is obtuse, the result can be adapted.

569. Let A, W, B, U, C, V be six points in this order on a circle such that AU, BV and CW all intersect in the common point P at angles of $60°$. Prove that
$$|PA| + |PB| + |PC| = |PU| + |PV| + |PW| \ .$$

Solution 1. [A. Abdi] We first recall the result: *Suppose that DEF is an equilateral triangle and that G is a point on the minor arc EF of the circumcircle of DEF. Then $|DG| = |EG| + |FG|$.* (Select H on DG so that $EH = EG$. Since $\angle EGH = 60°$, triangle EGH is equilateral. It can be shown that triangle DEH and FEG are congruent (SAS), so that $|DG| = |DH| + |HG| = |FE| + |EG|$.)

Let O be the centre of the circle and let K, M, N be respective feet of the perpendiculars from O to AU, BV, CW. Wolog, let K be between P and A, M between P and V and N be between P and C. Since triangles PKO, PMO and PNO are right with hypotenuse PO, the points O, P, K, M, N are all equidistant from the midpoint of OP and so are concyclic.

P and M lie on opposite arcs KN so $\angle NMK = 180° - \angle NPK = 180° - \angle CPA = 60°$. Also $\angle NKM = \angle NPM = 60°$ and $\angle KNM = \angle KPM = 60°$, so that triangle KMN is equilateral and $|PM| = |PK| + |PN|$.

Hence
$$\begin{aligned}
(|AP| + |BP| + |CP|) &- (|UP| + |VP| + |WP|) \\
&= (|AK| + |PK| + |BM| - |PM| + |CN| + |PN|) \\
&\quad - (|UK| - |PK| + |VM| + |PM| + |WN| - |PN|) \\
&= (|AK| - |UK|) + (|BM| - |VM|) + (|WN| - |PN|) + 2(|PK| - |PM| + |PN|) \\
&= 0 \ .
\end{aligned}$$

Comment. Several solvers tried the strategy of comparing the equation for two related positions, either with the situation where the second position put P at the centre of the circle, where the result is obvious, or moved P along one of the lines, say UA to a new position. In both case, the fact that the difference in the lengths of two parallel chords was split evenly to the two half chords played a role, as did the perpendiculars to the chords for one position of P from the other position of P.

Solution 2. [P.J. Zhao] Construct equilateral triangles BCD and VWT external to P. Then $PBDC$ and $PWTV$ are concyclic quadrilaterals so that $\angle DPC = \angle DBC = 60° = \angle UPC$ and $\angle TPV = \angle TWV = 60° = \angle APV$. Therefore, the points D, U, P, A, T are collinear.

Since $PD = PB + PC$ and $PT = PV + PW$ (see Solution 1), $|PA| + |PB| + |PC| = |DA|$ and $|PU| + |PV| + |PW| = |UT|$.

Let O be the centre of the circle. Triangles BDO and CDO are congruent (SSS), so that DO bisects angle BDC and so is perpendicular to BC. Similarly, $OT \perp VW$.

Let BC and VW intersect UA at E and S respectively. Then

$$\begin{aligned}\angle ODP &= 90^\circ - \angle CED = 90^\circ - \angle BEP \\ &= 90^\circ - (180^\circ - 60^\circ - \angle CBP) \\ &= 90^\circ - (180^\circ - 60^\circ - \angle VWP) \\ &= 90^\circ - \angle VSP = \angle OTP \ .\end{aligned}$$

Therefore, triangle DOT is isosceles and so $OD = OT$. Also $OU = OA$ and $\angle OUT = \angle OAD$. Therefore triangles DAO and TUO are congruent (ASA) and so $DA = UT$. Hence

$$|PA| + |PB| + |PC| = |PU| + |PV| + |PW| \ .$$

Solution 3. [J. Zung] Construct the equilateral triangles BCD and WVT and adopt the notation of Solution 2. Observe that P is the Fermat point of both triangles ABC and UVW; this is the point that minimizes the sum of the distances from P to the vertices of the triangle and is characterized as that point from which the rays to the vertices meet at an angle of 120°. This point has the property, that when an external equilateral triangle is erected on one side of the triangle, the line joining the vertices of the given triangle and equilateral triangle not on the common side passes through it. In the present situation, this implies that D, U, P, A, T are collinear.

Consider the rotation with centre D through an angle of 60° that takes $B \to C$, $C \to E$, $P \to Q$. Then

$$\begin{aligned}\angle QCP &= \angle QCE + \angle ECD + \angle DCB + \angle BCP \\ &= \angle PBC + 60^\circ + 60^\circ + \angle BCP = 180^\circ \ .\end{aligned}$$

Thus, Q, C, P are collinear. Since $\angle PDQ = 60^\circ$, triangle PDQ is equilateral, so that $|PQ| = |PD|$. Therefore

$$\begin{aligned}|PA| + |PB| + |PC| &= |PA| + |CQ| + |PC| \\ &= |PA| + |PQ| = |PA| + |DP| = |DA| \ .\end{aligned}$$

Similarly, $|PU| + |PV| + |PT| = |UT|$.

Let O be the centre of the circle. Since B, W, V, C are concyclic, $\angle BCW = \angle BVW$. Since $\angle BDC + \angle BPC = 180^\circ$, then B, D, C, P are concyclic and $\angle BDP = \angle BCD$. Since the right bisector of BC passes through D and O, $\angle BDO = 30^\circ$. Hence

$$\angle ODP = 30^\circ - \angle BDP = 30^\circ - \angle BCP = 30^\circ - \angle BCW \ .$$

Similarly, $\angle OTP = 30^\circ - \angle BVW$. Therefore $\angle ODP = \angle OTP$, triangle ODT is isosceles and so $DF = FT$, where F is the foot of the perpendicular from O to UA, Since, also, $FU = FA$, it follows that

$$|DA| = |DF| + |FA| = |FT| + |FU| = |UT|$$

and the desired result obtains.

Solution 4. [P. Wen] Let the centre of the circle be at the origin, the coordinates of P be (p, q) and the respective lengths of PA, PB, PC, PU, PV, PW be a, b, c, u, v, w. Take UA to be parallel to the $x-$axis. Then

$$A \sim (p + a, q) \qquad B \sim (p - b/2, q + b\sqrt{3}/2) \qquad C = (p - c/2, q - c\sqrt{3}/2)$$

$$U \sim (p-u, q) \qquad V \sim (p+v/2, q-v\sqrt{3}/2) \qquad W = (p+w/2, q+w\sqrt{3}/2) \ .$$

Since $AO = UO$,

$$\begin{aligned}(p+a)^2 + q^2 = (p-u)^2 + q^2 &\Longrightarrow a^2 + 2ap = u^2 - 2up \\ &\Longrightarrow 0 = (a+u)(a-u+2p) \Longrightarrow u = a + 2p \ .\end{aligned}$$

Since $BO = VO$,

$$\begin{aligned}(p-b/2)^2 + (q+b\sqrt{3}/2)^2 &= (p+v/2)^2 + (q-v\sqrt{3}/2)^2 \\ &\Longrightarrow b^2 - b(p-q\sqrt{3}) = v^2 + v(p-q\sqrt{3}) \\ &\Longrightarrow 0 = (b+v)(b-v-p+q\sqrt{3}) \Longrightarrow v = b - p + q\sqrt{3} \ .\end{aligned}$$

Since $CO = WO$,

$$c^2 - c(p+q\sqrt{3}) = w^2 + w(p+q\sqrt{3}) \Longrightarrow w = c - p - q\sqrt{3} \ .$$

Therefore $u + v + w = a + b + c$.

Solution 5. Let $|PA| = a$, $|PB| = b$, $|PC| = c$, $|PU| = u$, $|PV| = v$, $|PW| = w$. Let r be the radius and O the centre of the circle. Suppose that $|OP| = d$. Let A, W, B be on one side of OP and U, C, V be on the other side.

Let $\angle APO = \alpha \leq 60°$. Then $\angle WPO = \alpha + 60°$, $\angle BPO = \alpha + 120°$. $\angle UPO = 180° - \alpha$, $\angle CPO = 120° - \alpha$, $\angle VPO = 60° - \alpha$.

Using the Law of Cosines, we obtain that

$$\begin{aligned}r^2 &= a^2 + d^2 - 2ad\cos\alpha \\ &= w^2 + d^2 - 2wd\cos(\alpha + 60°) \\ &= b^2 + d^2 - 2bd\cos(\alpha + 120°) \\ &= u^2 + d^2 - 2ud\cos(180° - \alpha) = u^2 + d^2 + 2bd\cos\alpha \\ &= c^2 + d^2 - 2cd\cos(120° - \alpha) = c^2 + d^2 + 2cd\cos(\alpha + 60°) \\ &= v^2 + d^2 - 2vd\cos(60° - \alpha) = v^2 + d^2 + 2vd\cos(\alpha + 120°) \ .\end{aligned}$$

Each of these equations is a quadratic of the form

$$x^2 - (2d\cos\theta)x + (d^2 - r^2) = 0 \ .$$

It has one positive and one non-positive root. Since $r^2 - d^2\sin^2\theta \geq d^2\cos^2\theta$, the positive root is

$$\frac{2d\cos\theta + \sqrt{4d^2\cos^2\theta - 4d^2 + 4r^2}}{2} = d\cos\theta + \sqrt{r^2 - d^2\sin^2\theta} \ .$$

Hence,

$$a = d\cos\alpha + \sqrt{r^2 - d^2\sin^2\alpha} \ ;$$

$$b = d\cos(\alpha + 120°) + \sqrt{r^2 - d^2\sin^2(\alpha + 120°)} \ ;$$

$$c = -d\cos(\alpha + 60°) + \sqrt{r^2 - d^2\sin^2(\alpha + 60°)} \ ;$$

$$u = -d\cos\alpha + \sqrt{r^2 - d^2\sin^2\alpha} \ ;$$

$$v = -d\cos(\alpha + 120°) + \sqrt{r^2 - d^2\sin^2(\alpha + 120°)} \ ;$$

$$w = d\cos(\alpha + 60°) + \sqrt{r^2 - d^2\sin^2(\alpha + 60°)} \ ;$$

therefore

$$\begin{aligned}(a+b+c) - (u+v+w) &= 2d[\cos\alpha + \cos(\alpha + 120°) - \cos(\alpha + 60°)] \\ &= 2d[\cos\alpha(1 + \cos 120° - \cos 60°] - \sin\alpha(\sin 120° - \sin 60°)] = 0 \ ,\end{aligned}$$

as desired.