PROBLEMS FOR FEBRUARY

Please send your solutions to

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no later than March 5, 2009.

Electronic files can be sent to *barbeau@math.utoronto.ca*. However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

598. Let $a_1, a_2, \cdots, a_n$ be a finite sequence of positive integers. If possible, select two indices j, k with $1 \leq j < k \leq n$ for which a_j does not divide a_k; replace a_j by the greatest common divisor of a_j and a_k, and replace a_k by the least common multiple of a_j and a_k. Prove that, if the process is repeated, it must eventually stop, and the final sequence does not depend on the choices made.

599. Determine the number of distinct solutions x with $0 \leq x \leq \pi$ for each of the following equations. Where feasible, give an explicit representation of the solution.

(a) $8 \cos x \cos 2x \cos 4x = 1$;

(b) $8 \cos x \cos 4x \cos 5x = 1$.

600. Let $0 < a < b$. Prove that, for any positive integer n,

$$\frac{b+a}{2} \leq \sqrt[n]{\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}} \leq \sqrt[n]{\frac{a^n + b^n}{2}} .$$

601. A convex figure lies inside a given circle. The figure is seen from every point of the circumference of the circle at right angles (that is, the two rays drawn from the point and supporting the convex figure are perpendicular). Prove that the centre of the circle is a centre of symmetry of the figure.

602. Prove that, for each pair (m, n) of integers with $1 \leq m \leq n$,

$$\sum_{i=1}^{n} i(i-1)(i-2)\cdots(i-m+1) = \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} .$$

(b) Suppose that $1 \leq r \leq n$; consider all subsets with r elements of the set $\{1, 2, 3, \cdots, n\}$. The elements of this subset are arranged in ascending order of magnitude. For $1 \leq i \leq r$, let t_i denote the ith smallest element in the subset, and let $T(n, r, i)$ denote the arithmetic mean of the elements t_i. Prove that

$$T(n, r, i) = i\left(\frac{n+1}{r+1}\right) .$$

603. For each of the following expressions severally, determine as many integer values of x as you can so that it is a perfect square. Indicate whether your list is complete or not.

(a) $1 + x$;

(b) $1 + x + x^2$;

(c) $1 + x + x^2 + x^3$;
(d) $1 + x + x^2 + x^3 + x^4$;
(e) $1 + x + x^2 + x^3 + x^4 + x^5$.

604. $ABCD$ is a square with incircle Γ. Let l be a tangent to Γ, and let A', B', C', D' be points on l such that AA', BB', CC', DD' are all prependicular to l. Prove that $AA' \cdot CC' = BB' \cdot DD'$.

Note: there was an error in the statement of one of the December problems. I will state it as intended and repose it with this set. Solutions may be sent to E. Barbeau:

585. Calculate the number
$$b = \lfloor (\sqrt{n-1} + \sqrt{n} + \sqrt{n+1})^2 \rfloor \ .$$

Challenge Problems

I am starting new feature of Olymon, a list of problems that look interesting and appropriate for which I do not have a solution. They may turn out to be trivial, easy, difficult or impossible. There is no fixed deadline for their solution, but I will acknowledge successful solutions in the order in which I receive them. When I have a solution in hand, then I will add them to the regular Olymon stock for everyone to have a go at them.

C1. The function $f(x)$ is defined for real nonzero x, takes nonzero real values and satisfies the functional equation
$$f(x) + f(y) = f(xyf(x+y)) \ ,$$
whenever $xy(x+y) \neq 0$. Determine all possibilities for f.

C2. Let T be a triangle in the plane whose vertices are lattice points (*i.e.*, both coordinates are integers), whose edges contain no lattice points in their interiors and whose interior contains exactly one lattice point. Must this lattice point in the interior be the centroid of the T?

C3. Two circles are externally tangent at A and are internally tangent to a third circle Γ at points B and C. Suppose that D is the midpoint of the chord of Γ that passses through A and is tangent there to the two smaller given circles. Suppose, further, that the centres of the three circles are not collinear. Prove that A is the incentre of triangle BCD.

C4. Let a, b, c, m be positive integers for which $abcm = 1 + a^2 + b^2 + c^2$. Show that $m = 4$, and that there are actually possibilities with this value of m.

C5. Solve the equation
$$x^{12} - x^9 + x^4 - x = 1 \ .$$