PROBLEMS FOR MARCH

Please send your solutions to

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no later than April 15, 2009.

Electronic files can be sent to *rosumihai@yahoo.com*. However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

Problems for March, 2009

605. Prove that the number $299\cdots998200\cdots029$ can be written as the sum of three perfect squares of three consecutive numbers, where there are $n-1$ nines between the first 2 and the 8, and $n-2$ zeros between the last pair of twos.

606. Let $x_1 = 1$ and let $x_{n+1} = \sqrt{x_n^2 + n^2}$ for each positive integer n. Prove that the sequence $\{x_n : n > 1\}$ consists solely of irrational numbers and calculate $\sum_{k=1}^{n} \lfloor x_n^2 \rfloor$, where $\lfloor x \rfloor$ is the largest integer that does not exceed x.

607. Solve the equation

$$\sin x \left(1 + \tan x \tan \frac{x}{2}\right) = 4 - \cot x \ .$$

608. Find all positive integers n for which n, $n^2 + 1$ and $n^3 + 3$ are simultaneously prime.

609. The first term of an arithmetic progression is 1 and the sum of the first nine terms is equal to 369. The first and ninth terms of the arithmetic progression coincide respectively with the first and ninth terms of a geometric progression. Find the sum of the first twenty terms of the geometric progression.

610. Solve the system of equations

$$\log_{10}(x^3 - x^2) = \log_5 y^2$$

$$\log_{10}(y^3 - y^2) = \log_5 z^2$$

$$\log_{10}(z^3 - z^2) = \log_5 x^2$$

where $x, y, z > 1$.

611. The triangle ABC is isosceles with $AB = AC$ and I and O are the respective centres of its inscribed and circumscribed circles. If D is a point on AC for which $ID \| AB$, prive that $CI \perp OD$.

Challenge problems

The following problems are ones for which I do not have a solution. You are invited to solve them and send in your solution as early as you can. The first four problems posed last month have been solved, but I am still open to receiving further solutions. They will be posed at part of a future Olymon and I will acknowledge the solvers then.

C5. Solve the equation

$$x^{12} - x^9 + x^4 - x = 1 \ .$$

C6. Suppose that $n > 1$ and that S is the set of all polynomials of the form

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 \ ,$$

whose coefficients are complex numbers. Determine the minimum value over all such polynomials of the maximum value of $|p(z)|$ when $|z| = 1$.

C7. Let $a_1, a_2, \cdots, a_n$ be distinct integers. Prove that the polynomial

$$p(z) = (z - a_1)^2(z - a_2)^2 \cdots (z - a_n)^2 + 1$$

cannot be written as the product of two nonconstant polynomials with integer coefficients.

C8. Determine the locus of one focus of an ellipse reflected in a variable tangent to the ellipse.

C9. Let I be the centre of the inscribed circle of a triangle ABC and let u, v, w be the respective lengths of IA, IB, IC. Let P be any point in the plane and p, q, r the respective lengths of PA, PB, PC. Prove that, with the sidelengths of the triangle given conventionally as a, b, c,

$$ap^2 + bq^2 + cr^2 = au^2 + bv^2 + cw^2 + (a + b + c)z^2 \ ,$$

where z is the length of IP.