PROBLEMS FOR MAY 2009

Please send your solutions to

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individually as you solve the problems. Electronic files can be sent to *barbeau@math.utoronto.ca*. However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

New procedure. Instead of the monthly problem sets with a long deadline, I plan to send out one or two problems a week which should be solved as soon as you can. I will record the problems in order of receipt and acknowledge solvers. Solutions will be published when there is no more activity on a problem. This month, I will pose the challenge problems for which solutions have been received to give anyone else a chance to solve them before solutions appear.

For those of you who are looking for practice problems, you can access old Olymon problems and solutions on the website *www.math.utoronto.ca/barbeau/home.html* or *www.cms.math.ca*; on the CMS website, you can also access International Mathematical Talent Search Problems as well as problems posed on the Canadian Open Mathematics Challenge and the Canadian Mathematical Olympiad.

619. [Solved 2/3/09 by Jonathan Schneider] Suppose that $n > 1$ and that S is the set of all polynomials of the form

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 \ ,$$

whose coefficients are complex numbers. Determine the minimum value over all such polynomials of the maximum value of $|p(z)|$ when $|z| = 1$.

620. [Solved 2/3/09 by Jonathan Schneider, 4/3/09 by Cameron Bruggeman, 18/4/09 by Hao Sun, and 27/4/09 by Ahmad Abdi] Let $a_1, a_2, \cdots, a_n$ be distinct integers. Prove that the polynomial

$$p(z) = (z - a_1)^2 (z - a_2)^2 \cdots (z - a_n)^2 + 1$$

cannot be written as the product of two nonconstant polynomials with integer coefficients.

621. [Solved 2/3/09 by Jonathan Schneider and 27/4 by Ahmad Abdi] Determine the locus of one focus of an ellipse reflected in a variable tangent to the ellipse.

622. [Solved 10/4/09 by Robin Cheng] Let I be the centre of the inscribed circle of a triangle ABC and let u, v, w be the respective lengths of IA, IB, IC. Let P be any point in the plane and p, q, r the respective lengths of PA, PB, PC. Prove that, with the sidelengths of the triangle given conventionally as a, b, c,

$$ap^2 + bq^2 + cr^2 = au^2 + bv^2 + cw^2 + (a + b + c)z^2 \ ,$$

where z is the length of IP.

623. [Solved 12/4 by Jonathan Schneider] Given the parameters a, b, c, solve the system

$$x + y + z = a + b + c;$$

$$x^2 + y^2 + x^2 = a^2 + b^2 + c^2;$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3 \ .$$

624. [Solved 12/4/09 by Jonathan Schneider] Suppose that $x_i \geq 0$ and

$$\sum_{i=1}^{n} \frac{1}{1+x_i} \leq 1 \ .$$

Prove that

$$\sum_{i=1}^{n} 2^{-x_i} \leq 1 \ .$$

Problem of the Week for May 3 - 9.

625. Given an odd number of intervals, each of unit length, on the real line, let S be the set of numbers that are in an odd number of these intervals. Show that S is a finite union of disjoint intervals of total length not less than 1.

Challenge problem

C13. Determine all positive integers x and y for which $x^2 + 7 = 2^y$.