Solutions for May

619. Suppose that $n > 1$ and that S is the set of all polynomials of the form

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 \ ,$$

whose coefficients are complex numbers. Determine the minimum value over all such polynomials of the maximum value of $|p(z)|$ when $|z| = 1$.

Solution. [J. Schneider] For each value of n, the minimum is equal to 1. This minimum is attained for the polynomial z^n whose absolute value is equal to 1 when $|z| = 1$.

Let $q(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + 1$, so that $p(z) = z^n q(1/z)$. Hence $|p(z)| = |q(1/z)|$, when $|z| = 1$. Thus, the existence of z with $|z| = 1$ for which $|p(z)| \geq 1$ is equivalent to the existence of z with $|z| = 1$ for which $|q(z)| \geq 1$.

Let ζ be a primitive $(n+1)$th root of unity ($i.e.$, $\zeta = \cos(2\pi/(n+1)) + i\sin(2\pi/(n+1))$, say). Then the set of $(n+1)$ roots of unity consists of 1 and $\zeta_k = \zeta^k$ (for $1 \leq k \leq n$). Observe that for $1 \leq i \leq n$,

$$1 + \zeta_1^i + \zeta_2^i + \cdots + \zeta_n^i = 1 + (\zeta^i)^1 + (\zeta^i)^2 + \cdots + (\zeta^i)^n = \frac{(\zeta_i)^{n+1} - 1}{\zeta_i - 1} = 0 \ .$$

Therefore

$$q(1) + q(\zeta_1) + q(\zeta_2) + \cdots + q(\zeta_n) = a_0(1 + \zeta_1^n + \cdots + \zeta_n^n) + \cdots + a_{n-1}(1 + \zeta_1 + \cdots + \zeta_n) + (n+1) = n+1 \ .$$

However, then

$$n + 1 = |q(1) + q(\zeta_1) + \cdots + q(\zeta_n)| \leq |q(1)| + |q(\zeta_1)| + \cdots + |q(\zeta_n)| \ ,$$

so that at least one of the values in the right member is not less than 1. The desired result follows.

620. Let $a_1, a_2, \cdots, a_n$ be distinct integers. Prove that the polynomial

$$p(z) = (z - a_1)^2 (z - a_2)^2 \cdots (z - a_n)^2 + 1$$

cannot be written as the product of two nonconstant polynomials with integer coefficients.

Solution. Suppose, if possible that $p(z) = q(z)r(z)$, where $q(z)$ and $r(z)$ are two polynomials of positive degree with integer coefficients. Then, for each a_i, $q(a_i)$ and $r(a_i)$ are integers whose product is 1; therefore they can be only 1 or -1. Since the polynomial $p(z)$ is positive for real z, neither of the polynomials $q(z)$ nor $r(z)$ can vanish for any real value of z; therefore, the sign of each is constant for real z. By multiplying both by -1 if necessary, we may assume that both polynomials q and r are always positive for real z. Hence $q(a_i) = r(a_i) = 1$ for $1 \leq i \leq n$. Thus, each of the polynomial $q(z) - 1$ and $r(z) - 1$ has n distinct zeros a_i and so have degree not less than n. Since the degree of $p(z)$ is exactly $2n$, the degrees of $q(z)$ and $r(z)$ must be exactly n. Therefore

$$q(z) = r(z) = 1 + (z - a_1)(z - a_2)(z - a_3) \cdots (z - a_n) \ .$$

Therefore

$$(z - a_1)^2 (z - a_2)^2 (z - a_3)^2 \cdots (z - a_n)^2 + 1 = q(z)^2 \ ,$$

whence

$$1 = [q(z) - (z - a_1)(z - a_2) \cdots (z - a_n)][q(z) + (z - a_1)(z - a_2) \cdots (z - a_n)] \ .$$

But this is impossible as the second factor on the right has positive degree. The desired result follows.

621. Determine the locus of one focus of an ellipse reflected in a variable tangent to the ellipse.

Solution. Let the foci of the ellipse be F and G, and let P be an arbitrary point on the ellipse. Suppose that H is the reflected image of F in the tangent through P. We note that

$$|HP| + |GP| = |FP| + |GP|$$

is constant. Also, if X is an arbitrary point on the tangent on the same side of P as FH and Y is a point on the tangent on the opposite side, then $\angle HPX = \angle FPX = \angle GPY = 180^\circ - \angle GPX$, so that G, P, H are collinear. Therefore H lies on the circle with centre G and radius $|GP| + |FP|$.

Conversely, let K be any point on this circle. Since the ellipse is contained in the interior of the circle, the segment GK intersects the ellipse at a point P. We have that

$$|PK| = |GK| - |GP| = |FP| \ .$$

Let XY be the tangent to the ellipse at P with X on the same side of P as KF and Y on the opposite side. Then

$$\angle KPX = \angle GPY = \angle FPX \ ,$$

from which it follows that K is the reflection of F in the tangent XY.

Comment. To show that the locus is the prescribed circle, you need to show, not only that each point on the locus lies on the circle, but also that each point on the circle satisfies the locus condition.

622. Let I be the centre of the inscribed circle of a triangle ABC and let u, v, w be the respective lengths of IA, IB, IC. Let P be any point in the plane and p, q, r the respective lengths of PA, PB, PC. Prove that, with the sidelengths of the triangle given conventionally as a, b, c,

$$ap^2 + bq^2 + cr^2 = au^2 + bv^2 + cw^2 + (a+b+c)z^2 \ ,$$

where z is the length of IP.

Solution 1. [R. Cheng] The equation can be rearranged to read

$$a(p^2 - u^2 - z^2) + b(q^2 - v^2 - z^2) + c(r^2 - w^2 - z^2) = 0 \ .$$

By the Law of Cosines applied to triangle API, we have that

$$p^2 - u^2 - z^2 = 2uz \cos \angle PIA = \overrightarrow{IA} \cdot \overrightarrow{IP} \ .$$

Similar relations can be obtained for triangles PIB and PIC, and so the equation to be derived is

$$a\overrightarrow{IA} \cdot \overrightarrow{IP} + b\overrightarrow{IB} \cdot \overrightarrow{IP} + c\overrightarrow{IC} \cdot \overrightarrow{IP} = 0 \ .$$

Since this has to be derived for all points P, we need to show that

$$a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC} = \overrightarrow{O} \ .$$

We show that $a\overrightarrow{IA} + b\overrightarrow{IB}$ is collinear with $\overrightarrow{IC}$. Construct points X and Y on the line CI so that AX and BY are both perpendicular to CI. Let CI and AB intersect at Z. Then

$$\begin{aligned}
\angle XAI &= 90^\circ - \angle AIX = 90^\circ - (180^\circ - \angle IZA - \angle ZAI) \\
&= \angle IZA + \frac{1}{2}\angle BAC - 90^\circ \\
&= (180^\circ - \angle BAC - \frac{1}{2}\angle ACB) + \frac{1}{2}\angle BAC - 90^\circ \\
&= 90^\circ - \frac{1}{2}(\angle BAC + \angle ACB) \\
&= 90^\circ - \frac{1}{2}(\angle A + \angle C) = \frac{1}{2}\angle B \ .
\end{aligned}$$

Hence $|AX| = |AI|\cos(B/2)$. Similarly $|BY| = |BI|\cos(A/2)$. Then, from the Law of Sines, $AI : IB = \sin(B/2) : \sin(A/2)$ and $AX : BY = \sin(B/2)\cos(B/2) : \sin(A/2)\cos(A/2) = \sin B : \sin A$, from which $a|AX| = b|BY|$. Thus, $a\overrightarrow{AX} + b\overrightarrow{BY}$ has zero component in the direction orthogonal to CI and so $a\overrightarrow{IA} + b\overrightarrow{IB}$ is collinear with $\overrightarrow{IC}$. Repeat this for the other two vectors to find that $a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC} = 0$ is collinear with each of its summands, and therefore must be zero.

Solution 2. [N. Lvov] Let $\mathbf{p} = \overrightarrow{AP}$, $\mathbf{q} = \overrightarrow{BP}$, $\mathbf{r} = \overrightarrow{CP}$, $\mathbf{a} = \overrightarrow{BC}$, $\mathbf{b} = \overrightarrow{CA}$, $\mathbf{c} = \overrightarrow{AB}$ and $\mathbf{z} = \overrightarrow{IP}$. Let

$$\mathbf{u} = \frac{b\mathbf{c} - c\mathbf{b}}{a+b+c} \ .$$

This is a vector that points into the triangle from vertex A. Suppose that Q is the tip of this vector, so that $\mathbf{u} = \overrightarrow{AQ}$. The distance of Q from side AC is equal to

$$\frac{2[AQC]}{b} = \frac{|\mathbf{u} \times \mathbf{b}|}{b} = \frac{|\mathbf{b} \times \mathbf{c}|}{a+b+c} = \frac{2[ABC]}{a+b+c} \ ,$$

which is the inradius of triangle ABC. Similarly, the distance of Q from side AB is equal to the inradius. Therefore, Q must be the incentre of the triangle. A similar analysis can be made for the other two vertices of the triangle and we find that

$$\mathbf{u} = \frac{b\mathbf{c} - c\mathbf{b}}{a+b+c} = \overrightarrow{AI} \ ;$$

$$\mathbf{v} \equiv \frac{c\mathbf{a} - a\mathbf{c}}{a+b+c} = \overrightarrow{BI} \ ;$$

and

$$\mathbf{w} = \frac{a\mathbf{b} - b\mathbf{a}}{a+b+c} = \overrightarrow{CI} \ .$$

Since $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$,

$$a(\mathbf{p} + \mathbf{u}) + b(\mathbf{q} + \mathbf{v}) + c(\mathbf{r} + \mathbf{w}) = a(\mathbf{p} - \mathbf{u}) + b(\mathbf{q} - \mathbf{v}) + c(\mathbf{r} - \mathbf{w}) \ .$$

Taking the dot product of this equation with the vector $\mathbf{z} = \mathbf{p} - \mathbf{u} = \mathbf{q} - \mathbf{v} = \mathbf{r} - \mathbf{w}$ leads to

$$(ap^2 + bq^2 + cr^2) - (au^2 + bv^2 + cr^2) = (a+b+c)z^2 \ ,$$

as desired.

623. Given the parameters a, b, c, solve the system

$$x + y + z = a + b + c;$$

$$x^2 + y^2 + x^2 = a^2 + b^2 + c^2;$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3 \ .$$

Solution. [N. Lvov, J. Schneider] The first and third equations represent two planes in space that intersect in a line; the second represents a sphere, which the line intersects in at most two points. Therefore there are at most two solutions to the equation. One is $(x, y, z) = (a, b, c)$. The second is equal to

$$(x, y, z) = (a[1 - k(b-c)], b[1 - k(c-a)], c[1 - k(a-b)])$$

where

$$\begin{aligned} k &= \frac{2[a^2(b-c) + b^2(c-a) + c^2(a-b)]}{a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2} \\ &= \frac{(a-b)(b-c)(c-a)}{a^2b^2 + b^2c^2 + c^2a^2 - abc(a+b+c)} \ . \end{aligned}$$

Comment. This satisfies the linear equations regardless of the value of k, and substitution into the quadratic equation will establish the appropriate value of k.

624. Suppose that $x_i \geq 0$ and

$$\sum_{i=1}^{n} \frac{1}{1+x_i} \leq 1 \; .$$

Prove that

$$\sum_{i=1}^{n} 2^{-x_i} \leq 1 \; .$$

Solution. [J. Schneider] Let $f(x) = x2^{1/x}$. Since $f'(x) = (1 - (\log 2/x))2^{1/x} < 0$ for $0 < x < \log 2$, it follows that $f(x)$ decreases on the interval $(0, \frac{1}{2}]$.

The function 2^{x-1} is convex, so that the graphs of $y = x$ and $y = 2^{x-1}$ intersect in at most two points. Since they intersect at $x = 1$ and $x = 2$, it follows that $x > 2^{x-1}$ when $1 < x < 2$ and $x < 2^{x-1}$ when $x > 2$.

It suffices to prove the problem under the condition that $\sum (1 + x_i)^{-1} = 1$, for if $\sum (1 + x_i)^{-1} < 1$, then we can select $X > 0$ so that $(1 + X)^{-1} + \sum (1 + x_i)^{-1} = 1$ and obtain $2^{-X} + \sum 2^{-x_i} \leq 1$, from which the desired result would follow.

Let $y_i = (1 + x_i)^{-1}$ so that $\sum y_i = 1$. Suppose, to begin with that $y_i \leq \frac{1}{2}$ for each i. Then, since $f(y_i) \geq f(\frac{1}{2}) = 2$, it follows that

$$2^{-x_i} = 2^{(1-(1/y_i))} = \frac{2}{2^{1/y_i}} \leq y_i$$

so that $\sum_{i=1}^{n} 2^{-x_i} \leq \sum_{i=1}^{n} y_i = 1$ as desired.

The remaining case is that at least one y_i exceeds $\frac{1}{2}$. There can be at most one such y_i, so we may suppose that $y_1, y_2, \cdots, y_{n-1} \leq \frac{1}{2} < y_n$.

Suppose that $g(x) = 2^{(1-(1/x))}$. We show that

$$g(y_1) + g(y_2) + \cdots + g(y_{n-1}) \leq g(y_1 + y_2 + \cdots + y_{n-1}) \; .$$

Suppose that $Y = y_1 + y_2 + \cdots + y_{n-1}$; note that $Y < \frac{1}{2}$. Then

$$\begin{aligned} g(y_1) + g(y_2) + \cdots + g(y_n) &= 2\left[\frac{y_1}{f(y_1)} + \frac{y_2}{f(y_2)} + \cdots + \frac{y_{n-1}}{f(y_{n-1})}\right] \\ &\leq 2\left[\frac{y_1}{f(Y)} + \frac{y_2}{f(Y)} + \cdots + \frac{y_{n-1}}{f(Y)}\right] \\ &\leq \frac{2Y}{f(Y)} = g(Y) = g(y_1 + y_2 + \cdots + y_{n-1}) \; . \end{aligned}$$

We need to show that $\sum_{i=1}^{n} g(y_i) \leq 1$ when $\sum_{i=1}^{n} y_i = 1$. This can be achieved by showing that $g(Y) + g(1 - Y) \leq 1$; this amounts to

$$\frac{1}{2^{\frac{1-Y}{Y}}} + \frac{1}{2^{\frac{Y}{1-Y}}} \leq 1,$$

for $0 < Y < 1$. Let $z = (1 - Y)/Y$. Then we need to show that

$$\frac{1}{2^z} + \frac{1}{2^{1/z}} \leq 1$$

for $z > 0$. Since the left side takes the same value at z and $1/z$, it is enough to establish this for $z \geq 1$.

When $z \geq 2$, we can use the fact that $2^{z-1} \geq 2$ and Bernoulli's inequality to obtain

$$\left(1 - \frac{1}{2^z}\right)^z \geq 1 - \frac{z}{2^z} \geq 1 - \frac{1}{2} = \frac{1}{2} ,$$

from which $1 - 2^{-z} \geq 2^{-1/z}$ as desired.

Suppose that $1 \leq z \leq 2$. Let $h(z) = 2^{-z} + 2^{-1/z}$. Then $h(1) = 1$. We show that $h(z)$ decreases for $z \geq 1$.

$$h'(z) = -\log 2 \cdot 2^{-z} + \log 2 \cdot z^{-2} 2^{-1/z} .$$

Since $1 \leq z \leq 2$, we have that $z \geq 2^{z-1}$, so that $z^2 \geq 2^{2z-2}$. However

$$(2z - 2) - \left(z - \frac{1}{z}\right) = \left(z + \frac{1}{z}\right) - 2 \geq 0$$

so that $2z - 2 \geq z - (1/z)$. Therefore $z^2 \geq 2^{z - \frac{1}{z}}$ and so

$$h'(z) \leq -\log 2 \cdot 2^{-z} + \log 2 \cdot 2^{-1/z} \cdot 2^{\frac{1}{z} - z} = \log 2(-2^{-z} + 2^{-z}) = 0 .$$

Thus, $h(z)$ decreases on $[1, 2]$ and so $h(z) \leq 1$ there. This completes the solution.

625. Given an odd number of intervals, each of unit length, on the real line, let S be the set of numbers that are in an odd number of these intervals. Show that S is a finite union of disjoint intervals of total length not less than 1.

Solution. The result holds when there is one interval. Suppose that n is an odd number greater than 1 and, as an induction hypothesis, that the result holds for any odd number of intervals fewer than n. Since all of the intervals have the same length, they can be linearly ordered from left to right. Let Z be the rightmost interval and Y the next to rightmost interval. Let T be the union of all the intervals but Y and Z, and S' the set of points that belong to an odd number of the intervals making up T. By the induction hypothesis, S' is the union of a finite number of disjoint intervals not less than 1.

S contains the entire interval $Z \backslash Y$, as points here are contained only in Z; $S \cap (Y \cap Z) = S' \cap (Y \cap Z)$, as we are adding evenly many intervals to the collection making up T for the points in $Y \cap Z$. Thus, the only points that lie in S' but not in S must lie within $Y \backslash Z$. Note that these points deleted from S' constitute a union of intervals, since they are obtained by intersecting intervals. Since Y and Z have equal length, $|Y \backslash Z| = |Z \backslash Y|$ and so we augment S' by an interval that exceeds the length of the intervals of S' deleted. Therefore, the total length of the intervals making up S is at least 1.