Solutions for February

661. Let P be an arbitrary interior point of an equilateral triangle ABC. Prove that

$$|\angle PAB - \angle PAC| \geq |\angle PBC - \angle PCB| \quad .$$

Solution. The result is clear if P is on the bisector of the angle at A, since both sides of the inequality are 0.

Wolog, let P be closer to AB than AC, and let Q be the image of P under reflection in the bisector of the angle A. Then

$$\angle PAQ = \angle PAC - \angle QAC = \angle PAC - \angle PAB$$

and

$$\angle PCQ = \angle QCB - \angle PCB = \angle PBC - \angle PCB \ .$$

Thus, it is required to show that $\angle PAQ \geq \angle PCQ$.

Produce PQ to meet AB in R and AC in S. Consider the reflection $\mathfrak{R}$ with axis RS. The circumcircle $\mathfrak{C}$ of ΔARS is carried to a circle $\mathfrak{C}'$ with chord RS. Since $\angle RCS < 60° = \angle RAS$ and the angle subtended at the major arc of $\mathfrak{C}'$ by RS is $60°$, the point C must lie outside of $\mathfrak{C}'$. The circumcircle $\mathfrak{D}$ of ΔAPQ is carried by $\mathfrak{R}$ to a circle $\mathfrak{D}'$ with chord PQ. Since $\mathfrak{D}$ is contained in $\mathfrak{C}$, $\mathfrak{D}'$ must be contained in $\mathfrak{C}'$, so C must lie outside of $\mathfrak{D}'$. Hence $\angle PCQ$ must be less than the angle subtended at the major arc of $\mathfrak{D}'$ by PQ, and this angle is equal to $\angle PAQ$. The result follows.

662. Let n be a positive integer and $x > 0$. Prove that

$$(1+x)^{n+1} \geq \frac{(n+1)^{n+1}}{n^n} x \quad .$$

Solution 1. By the Arithmetic-Geometric Means Inequality, we have that

$$\frac{1+x}{n+1} = \frac{n(1/n)+x}{n+1} \geq \left[\left(\frac{1}{n}\right)^n x\right]^{\frac{1}{n+1}}$$

so that

$$\frac{(1+x)^{n+1}}{(n+1)^{n+1}} \geq \frac{x}{n^n}$$

and the result follows.

Solution 2. (by calculus) Let

$$f(x) = n^n(1+x)^{n+1} - (n+1)^{n+1}x \quad \text{for} \quad x > 0 \ .$$

Then

$$f'(x) = (n+1)[n^n(1+x)^n - (n+1)^n] = (n+1)n^n[(1+x)^n - (1+\frac{1}{n})^n]$$

so that $f'(x) < 0$ for $0 < x < 1/n$ and $f'(x) > 0$ for $1/n < x$. Thus $f(x)$ attains its minimum value 0 when $x = 1/n$ and so $f(x) \geq 0$ when $x > 0$. The result follows.

Solution 3. (by calculus) Let $g(x) = (1+x)^{n+1}x^{-1}$. Then $g'(x) = (1+x)^n x^{-2}[nx-1]$, so that $g(x) < 0$ for $0 < x < 1/n$ and $g'(x) > 0$ for $x > 1/n$. Therefore $g(x)$ assumes its minimum value of $(n+1)^{n+1}n^{-n}$ when $x = 1/n$, and the result follows.

Solution 4. [G. Ghosn] We make the substituion $t = (nx)^{1/(n+1)} \Leftrightarrow x = t^{n+1}/n$. Then it is required to prove that

$$1 + \frac{t^{n+1}}{n} \geq \frac{(n+1)t}{n} \ .$$

Observe that

$$\begin{aligned} t^{n+1} - (n+1)t - n &= t(t^n - 1) - n(t-1) = (t-1)(t^n + t^{n-1} + \cdots + t - n) \\ &= (t-1)[(t^n - 1) + (t^{n-1} - 1) + \cdots + (t-1)] \\ &= (t-1)^2[t^{n-1} + 2t^{n-2} + \cdots + (n-1)] \geq 0 \ , \end{aligned}$$

for $t > 0$. The desired result follows.

Solution 5. Let $u = nx - 1$ so that $x = (1+u)/n$. Then

$$\begin{aligned} (1+x)^{n+1} - \frac{(n+1)^{n+1}}{n^n}x &= (1 + \frac{1}{n} + \frac{u}{n})^{n+1} - (1 + \frac{1}{n})^{n+1}(1+u) \\ &= (1+\frac{1}{n})^{n+1} + (n+1)(1+\frac{1}{n})^n\frac{u}{n} + \binom{n+1}{2}(1+\frac{1}{n})^{n-1}(\frac{u}{n})^2 \\ &\quad + \binom{n+1}{3}(1+\frac{1}{n})^{n-2}(\frac{u}{n})^3 + \cdots - (1+\frac{1}{n})^{n+1}(1+u) \\ &= \binom{n+1}{2}(1+\frac{1}{n})^{n-1}(\frac{u}{n})^2 + \binom{n+1}{3}(1+\frac{1}{n})^{n-2}(\frac{u}{n})^3 + \cdots \ . \end{aligned}$$

This is clearly nonnegative when $u \geq 0$. Suppose that $-1 < u < 0$. For $1 \leq k \leq n/2$, we have that

$$\binom{n+1}{2k}(1+\frac{1}{n})^{n-2k+1}(\frac{u}{n})^{2k} + \binom{n+1}{2k+1}(1+\frac{1}{n})^{n-2k}(\frac{u}{n})^{2k+1}$$

$$= \frac{(n+1)!(1+1/n)^{n-2k}}{(2k+1)!(n+1-2k)!}\left(\frac{u}{n}\right)^{2k}[(2k+1)(1+\frac{1}{n}) + (n+1-2k)(\frac{u}{n})] \ .$$

This will be nonnegative if and only if the quantity in square brackets is nonnegative. Since $u > -1$, this quantity exceeds

$$(2k+1)(1+\frac{1}{n}) - (n+1-2k)(\frac{1}{n}) = \left(\frac{n+1}{n}\right)(2k+1-1) - \frac{2k}{n} = 2k > 0 \ .$$

Thus, each consecutive pair of terms in the sequence

$$\binom{n+1}{2}(1+\frac{1}{n})^{n-1}(\frac{u}{n})^2 + \binom{n+1}{3}(1+\frac{1}{n})^{n-2}(\frac{u}{n})^3 + \cdots$$

has a positive sum and so the desired result follows.

663. Find all functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that

$$x^2y^2(f(x+y) - f(x) - f(y)) = 3(x+y)f(x)f(y)$$

for all real numbers x and y.

Solution. An obvious solution if $f(x) \equiv 0$. We consider other possibilities.

Setting $y = 0$ yields that $0 = 3xf(x)f(0)$ for all x. Setting $y = -x$ yields that $x^4[f(0) - f(x) - f(-x)] = 0$, so that $f(0) = f(x) + f(-x)$ for all nonzero x. Suppose, if possible, that $f(0) \neq 0$. Then, if $x \neq 0$, we must have that $f(x) = 0$, so that $f(0) = f(x) + f(-x) = 0$, a contradiction. Therefore, $f(0) = 0$ and so $f(x) = -f(-x)$ for all real x.

Setting $y = x$ yields that

$$f(2x) = \frac{6}{x^3} f(x)^2 + 2f(x)$$

for all nonzero x, while the sum $x = 2x + (-x)$ leads to

$$4x^4[2f(x) - f(2x)] = 3xf(2x)f(-x) = -3xf(2x)f(x) \ .$$

Therefore

$$4x^3 \left[\frac{6}{x^3} f(x)^2\right] = 3\left[\frac{6}{x^3} f(x)^2 + 2f(x)\right] f(x)$$

so that

$$8x^3 f(x)^2 = 6f(x)^3 + 2x^3 f(x)^2$$

or

$$f(x)^3 = x^3 f(x)^2 \ .$$

Therefore, for each real x, either $f(x) = 0$ or $f(x) = x^3$.

Suppose that $f(z) = 0$ for some real z; note that $y \neq 0$. Select x so that $f(x) \neq 0$ and let $y = z - x$. Then, since $x^2y^2[-f(x) - f(y)] = 3zf(x)f(y)$, $f(y) \neq 0$. Thus $f(x) = x^3$, $f(y) = y^3$ so that

$$-x^2y^2(x^3 + y^3) = 3(x + y)x^3y^3 \ .$$

This simplifies to

$$0 = x^2y^2(x + y)(x^2 + 2xy + y^2) = x^2y^2(x + y)^3$$

with the result that $z = x + y = 0$. Therefore $f(x) = x^3$ for all real x (including 0).

Comment. J. Seaton deserves credit for the argument that, if $f(x) = 0$ for all nonzero x, then $f(0) = 0$ as well.

664. The real numbers x, y, and z satisfy the system of equations

$$x^2 - x = yz + 1;$$

$$y^2 - y = xz + 1;$$

$$z^2 - z = xy + 1.$$

Find all solutions (x, y, z) of the system and determine all possible values of $xy + yz + zx + x + y + z$ where (x, y, z) is a solution of the system.

Solution. First we dispose of the situation that not all the variables takes distinct values. If $x = y = z$, then the equations reduce to $x = -1$, so that $(x, y, z) = (-1, -1, -1)$ is a solution and $x+y+z+xy+yz+zx = 0$.

By subtracting equations in pairs, we find that

$$0 = (x - y)(x + y + z - 1) = (y - z)(x + y + z - 1) = (z - x)(x + y + z - 1) \ .$$

Suppose that $x \neq y = z$. Then we must have $x + 2y = 1$ and $x^2 - x = y^2 + 1$, so that $0 = 3y^2 - 2y - 1 = (3y + 1)(y - 1)$. This leads to the two soutions $(x, y, z) = (-1, 1, 1), (\frac{5}{3}, -\frac{1}{3}, -\frac{1}{3})$. Symmetric permutations of these also are solutions and we find that $x + y + z + xy + yz + zx = 0$.

Henceforth, assume that the values of x, y, z are distinct. Any solution x, y, z of the system must satisfy the cubic equation

$$t^3 - t^2 - t = xyz \ .$$

In particular, from the coefficients, we find that $x+y+z=1$ and $xy+yz+zx=-1$ whence $xy+yz+zx+x+y+z=1$.

Conversely, suppose that we take any real number w. Let x, y, z be the roots of the cubic equation

$$t^3 - t^2 - t = w \ .$$

Then $xyz = w$. If $w = 0$, then the cubic equation has the roots $\{0, \frac{1}{2}(1+\sqrt{5}), \frac{1}{2}(1-\sqrt{5})\}$ and it can be checked that assigning these as the values of x, y and z any order will yields a solution to the given equation. If $w \neq 0$, then plugging the roots into the equation and dividing by it will yield the given system.

All that remains is to discover which values of w will yield three real roots for the cubic. Let $f(t) = t^3 - t^2 - t$. This function assumes a maximum value of $5/27$ at $t = -1/3$ and a minimum value of -1 when $t = 1$. Thus $f(t)$ assumes each value in the closed interval $[-1, 5/27]$ three times, counting multiplicity, and each other real value exactly once.

Thus, the solutions of the system are the roots of the cubic equation $t^3 - t^2 - t = w$, where w is any real number selected from the interval $[-1, 5/27]$.

(Note, that the "extreme" solutions are $(x,y,z) = (1,1,-1), (-1/3,-1/3,5/3)$. The only solution not related to the cubic is $(x,y,z) = (-1,-1,-1)$.)

Comment. G. Ajjanagadde, in the case of distinct values of x, y and z, obtained the equations $x+y+z = 1$ and $xy+yz+zx = -1$, whence, for given value of x, we get the system $y+z = 1-x$ and $yz = x^2-x-1$, so that y and z are solutions of the quadratic equation

$$t^2 - (1-x)t + (x^2 - x - 1) = 0 \ .$$

The discriminant of this quadratic is

$$(1-x)^2 - 4(x^2 - x - 1) = -3x^2 + 2x + 5 = -(3x-5)(x+1) \ .$$

Thus, we will obtain real values of x, y, z if and only if x, y and z lies between -1 and $5/3$ inclusive.

665. Let $f(x) = x^3 + ax^2 + bx + b$. Determine all integer pairs (a,b) for which $f(x)$ is the product of three linear factors with integer coefficients.

Solution. If $b = 0$, then the polynomial becomes $x^2(x+a)$, which satisfies the condition for all values of a. This covers the situation for which x is a factor of the polynomial. Since the leading coefficient of $f(x)$ is 1, the same must be true (up to sign) of its factors. Assume that $f(x) = (x+u)(x+v)(x+w)$ for integers u, v and w with $uvw \neq 0$. Since $uvw = uv + vw + wu = b$,

$$\frac{1}{u} + \frac{1}{v} + \frac{1}{w} = 1 \ .$$

It is clearly not possible for all of u, v and w to be negative. Nor can it occur that two of them, say v and w can be negative, for then the left side would be less than $1/u \leq 1$. Suppose that u and v are positive, while w is negative. One possibility is that $u = 1$ and $v = -w$ in which case $f(x) = (x+1)(x^2 - v^2) = x^3 + x^2 - v^2x - v^2$. If neither u nor v is equal to 1, then $1/u + 1/v + 1/w < 1/u + 1/v \leq 1$, and this case is not possible. Finally, suppose that u, v and w are all positive, with $u \leq v \leq w$. Then $1 \leq 3/u$, so that $u \leq 3$. A little trial and error leads to the possibilities $(u,v,w) = (3,3,3), (2,4,4)$ and $(2,3,6)$. Thus the possibilities for (a,b) are $(u,0)$, $(1,-v^2)$, $(9,27)$, $(10,32)$ and $(11,36)$. Indeed, $x^3+9x^2+27x+27 = (x+3)^3$, $x^3 + 10x^2 + 32x + 32 = (x+2)(x+4)^2$ and $x^3 + 11x^2 + 36x + 36 = (x+2)(x+3)(x+6)$.

666. Assume that a face S of a convex polyhedron $\mathfrak{P}$ has a common edge with every other face of $\mathfrak{P}$. Show that there exists a simple (nonintersecting) closed (not necessarily planar) polygon that consists of edges of $\mathfrak{P}$ and passes through all the vertices.

Solution. Suppose that the face S has m vertices A_1, A_2, $\cdots$, A_m listed in order, and that there are n vertices of $\mathfrak{P}$ not contained in S. We prove the result by induction on n. If $n = 1$, then every face abutting S is a triangle. Let X be the vertex off S; then $A_1 \cdots A_m X A_1$ is a polygonal path of the desired type. Suppose that the result holds for any number of vertices m of S and for n vertices off S where $1 \leq n \leq k$. Consider the case $n = k + 1$.

Consider the graph G of all vertices of $\mathfrak{P}$ and those edges of $\mathfrak{P}$ not bounding S. Since there are no faces bounded solely by these edges, the graph must be a *tree* (*i.e.*, it contains no loops and there is a unique path joining any pair of points). We show that there is at least one vertex X not in S for which every edge but one must connect X to a vertex of S. Suppose otherwise. Then, let us start with such a vertex X and form a sequence $X_1, X_2, \cdots$ of vertices not in S such that $X_i X_{i+1}$ are edges of $\mathfrak{P}$. Since the number of vertices off S is finite, there must be $i < j$ for which $X_i = X_j$ so that $X_i X_{i+1} \cdots X_{j-1} X_j$ is a loop in G. But this contradicts the fact that G is a tree.

Hence there is a vertex X with at most one adjacent edge not connecting it to S. If there were no such edge, then X would be the only vertex not in S, contradicting $k + 1 \geq 2$. Hence there is a vertex Y not in S such that XY is an edge of $\mathfrak{P}$. We may assume that Y is further from the plane of S than S. (If not, suppose that S is in the plane $z = 0$ and that $\mathfrak{P}$ lies in the quadrant $z > 0$, $y > 0$ with Y further than X from the plane $y = 0$. We can transform $\mathfrak{P}$ by a mapping of the type $(x, y, z) \rightarrow (x, y, z + \lambda y)$ for suitable positive λ. This will not alter the configuration of vertices and edges.) Extend YX to a point Z in the plane of S. Let $\mathfrak{Q}$ be the convex hull of (smallest closed convex set containing) Z and $\mathfrak{P}$. This will have a side T containing S of the form $A_1 A_2 \cdots A_r Z A_s \cdots A_m$ where $r < s$. The triangles XZA_r and XZA_s will be coplanar with faces of $\mathfrak{P}$, and the convex hull will have at most k vertices not on T. Every face of $\mathfrak{Q}$ will abut T. By the induction hypothesis, we can construct a polygon containing each vertex of $\mathfrak{Q}$. If an edge of this polygon is YZ and so includes X, and if one edge is say ZA_r, then we can replace these two edges by $YXA_sA_{s-1} \cdots A_{r+1}A_r$. If YZ is not an edge of this polygon, but A_rZ and ZA_s are, then we can replace these edges by $A_rXA_{r+1} \cdots A_s$. In both cases, we obtain a polygon of the required type for $\mathfrak{P}$.

667. Let A_n be the set of mappings $f : \{1, 2, 3, \cdots, n\} \longrightarrow \{1, 2, 3, \cdots, n\}$ such that, if $f(k) = i$ for some i, then f also assumes all the values $1, 2, \cdots, i-1$. Prove that the number of elements of A_n is $\sum_{k=0}^{\infty} k^n 2^{-(k+1)}$.

Solution 1. Let $u_0 = 1$ and, for $n \geq 1$, let u_n be the number of elements in A_n. Let $1 \leq r \leq n$. Consider the set of mappings in A_n for which the value 1 is assumed exactly r times. Then $1 \leq r \leq n$. Then each such mapping takes a set of $n - r$ points *onto* a set of the form $\{2, 3, \cdots, s\}$ where $s - 1 \leq n - r \leq n - 1$. Hence, there are u_{n-r} such mappings. Since there are $\binom{n}{r}$ possible sets on which a mapping may assume the value 1 r times,

$$u_n = \sum_{r=1}^{n} \binom{n}{r} u_{n-r} = \sum_{r=0}^{n-1} \binom{n}{r} u_r \ .$$

Now $u_0 = 1 = \sum_{k=0}^{\infty} 1/2^{k+1}$. Assume, as an induction hypothesis, that $u_r = \sum_{k=0}^{\infty} k^r/2^{k+1}$ for $0 \leq r \leq n-1$. Then

$$\begin{aligned} u_n &= \sum_{r=0}^{n-1} \binom{n}{r} u_r = \sum_{r=0}^{n-1} \binom{n}{r} \sum_{k=0}^{\infty} \frac{k^r}{2^{k+1}} \\ &= \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{r=0}^{n-1} \binom{n}{r} k^r = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} [(1+k)^n - k^n] \\ &= \sum_{k=0}^{\infty} \frac{(1+k)^n}{2^{k+1}} - \sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{k^n}{2^k} - \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}} \\ &= \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}} \end{aligned}$$

and the result follows. (The interchange of the order of summation and rearrangement of terms in the infinite sum can be justified by the absolute convergence of the series.)

Solution 2. For $1 \leq i$, let v_i be the number of mappings of $\{1, 2, \cdots, n\}$ *onto* a set of exactly i elements. Observe that $v_i = 0$ when $i \geq n+1$. There are k^n mappings of $\{1, 2, \cdots, n\}$ into $\{1, 2, \cdots, k\}$, of which v_k belong to A_n. The other $k^n - v_k$ mappings will leave out i numbers in the range for some $1 \leq i \leq k-1$, and the i numbers not found can be selected in $\binom{k}{i}$ ways. Thus

$$k^n = \sum_{i=1}^{k} \binom{k}{i} v_i \quad .$$

Hence

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} &= \sum_{k=0}^{\infty} \sum_{i=1}^{k} \frac{\binom{k}{i} v_i}{2^{k+1}} = \sum_{k=0}^{\infty} \sum_{i=1}^{n} \frac{\binom{k}{i} v_i}{2^{k+1}} \\ &= \sum_{i=1}^{n} \left(\sum_{k=0}^{\infty} \frac{\binom{k}{i}}{2^{k+1}} \right) v_i = \sum_{i=1}^{n} \left(\sum_{k=i}^{\infty} \frac{\binom{k}{i}}{2^{k+1}} \right) v_i \ .\end{aligned}$$

We evaluate the inner sum. Fix the positive integer i. Suppose that we flip a fair coin an indefinite number of times, and consider the event that the $(i+1)$th head occurs on the $(k+1)$th toss. Then the previous i heads could have occurred in $\binom{k}{i}$ posible positions, so that the probability of the event is $\binom{k}{i} 2^{-(k+1)}$. Since the $(i+1)$th head must occur on *some* toss with probability 1, $\sum_{k=i}^{\infty} \binom{k}{i} 2^{-(k+1)} = 1$. Hence

$$\sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} = \sum_{i=1}^{n} v_i = \# A_n \ .$$

Solution 3. [C. Deng] Let $s_n = \sum_{k=0}^{\infty} k^n 2^{-(k+1)}$; note that $s_0 = s_1 = 1$. Let $w_0 = 1$ and $w_n = \# A_n$ for $n \geq 1$, so that, in particular, $w_1 = 1$.

For $n \geq 0$,

$$\begin{aligned} s_{n+1} = 2s_{n+1} - s_{n+1} &= 2 \sum_{k=0}^{\infty} k^{n+1} 2^{-(k+1)} - \sum_{k=0}^{\infty} k^{n+1} 2^{-(k+1)} \\ &= \sum_{k=0}^{\infty} [(k+1)^{n+1} - k^{n+1}] 2^{-(k+1)} \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n+1}{i} k^i \right) 2^{-(k+1)} \\ &= \sum_{i=0}^{n} \left(\sum_{k=0}^{\infty} \binom{n+1}{i} k^i 2^{-(k+1)} \right) \\ &= \sum_{i=0}^{n} \binom{n+1}{i} s_i \ . \end{aligned}$$

We now show that w_n satisfies the same recursion. Suppose that g is an arbitrary element of A_{n+1} and that its maximum appears $n+1-i$ times, where $0 \leq i \leq n$. Then there are $\binom{n+1}{i}$ ways to choose the i remaining slots to fill with numbers without leaving gaps in the range, and then we can fill in the remaining $n+1-i$ slots with one more than the largest number in the range of the i slots. Thus, we find that $w_{n+1} = \sum_{i=0}^{n} \binom{n+1}{i} w_i$. The desired result now follows, since $s_0 = w_0$.