(solutions follow)

1998-1999 Olympiad Correspondence Problems

Set 4

19. The following statics problems involving vectors was given and two separate solutions provided. Determine whether either of them is correct and explain the discrepancy between the answers.

Problem. A force $\mathbf{R}$ of magnitude 200 N (Newtons) is the resultant of two forces $\mathbf{F}$ and $\mathbf{G}$ for which $2|\mathbf{F}| = 3|\mathbf{G}|$ and the angle between the resultant and $\mathbf{G}$ is twice the angle between the resultant and $\mathbf{F}$. Determine the magnitudes of $\mathbf{F}$ and $\mathbf{G}$.

Both solutions use the parallelogram representation of the vectors as illustrated, where $3u = |\mathbf{F}|$, $2u = |\mathbf{G}|$ and $v = |\mathbf{R}| = 200$. From the Law of Sines, we have that

$$\frac{\sin 2\theta}{3} = \frac{\sin \theta}{2}$$

whence $\cos\theta = 3/4$, $\cos 2\theta = 1/8$ and $\cos 3\theta = 4\cos^3\theta - 3\cos\theta = -9/16$. ¿From here, there are two ways to proceed:

(i) From the Law of Cosines, we find that

$$v^2 = 4u^2 + 9u^2 - 12u^2\cos(180^\circ - 3\theta) = 13u^2 + 12u^2\cos 3\theta = \frac{25u^2}{4}$$

so that $200 = \frac{5}{2}u$, $u = 80$, $|\mathbf{F}| = 240$ and $|\mathbf{G}| = 160$.

(ii) From the Law of Cosines, we find that

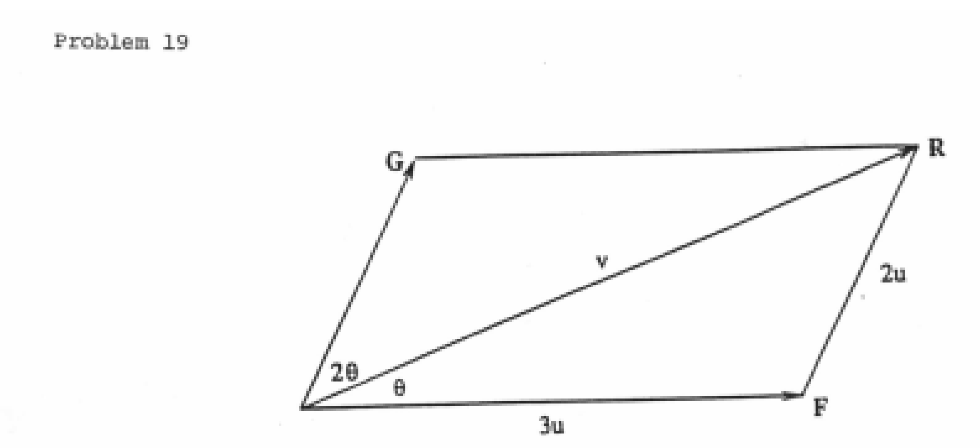
$$4u^2 = 9u^2 + v^2 - 6uv\cos\theta = 9u^2 + 200^2 - 900u$$

so that

$$0 = 5(u^2 - 180u + 8000) = 5(u - 80)(u - 100) \ .$$

Hence, $u = 80$, $|\mathbf{F}| = 240$, $|\mathbf{G}| = 160$ or $u = 100$, $|\mathbf{F}| = 300$, $|\mathbf{G}| = 200$.

Why does method (i) lead to one solution while method (ii) yields two? **QED**



20. Using a pair of compasses with a fixed radius exceeding half the length of a line segment AB, it is possible to determine a point C for which the triangle ABC is equilateral. Here is how it is done.

With centres A and B construct circles using the compasses and let P be one of the two points of intersection of these circles. Draw the circle with centre P, and let it intersect the circle of centre A in Q and the circle of centre B in R. There are different possible configurations, but we will select one so that QR is parallel to AB. Now construct, using the compasses again, circles of centres Q and R. These will intersect in P and a second point C. Prove that triangle ABC is equilateral.

21. In the following problem, we will begin by making some empirical observations. Your task will be to formulate some general results exemplified by them and provide proofs. You may observe some other general results not really pointed to below; if so, formulate and justify these.

The numbers $1, 2, 3, \cdots$ are placed in a triangular array and certain observations concerning row sums are made as indicated below:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 2 & & 3 & & & & \\
 & & & 4 & & 5 & & 6 & & & \\
 & & 7 & & 8 & & 9 & & 10 & & \\
 & 11 & & 12 & & 13 & & 14 & & 15 & \\
16 & & 17 & & 18 & & 19 & & 20 & & 21
\end{array}$$

$$1 = (0 + 1)(0^2 + 1^2)$$
$$5 = 1^2 + 2^2$$
$$15 = (1 + 2)(1^2 + 2^2)$$
$$34 = 2 \times (1^2 + 4^2)$$
$$65 = (2 + 3)(2^2 + 3^2)$$
$$111 = 3 \times (1^2 + 6^2)$$

$$1 = 1^4 \qquad 1 + 15 = 2^4 \qquad 1 + 15 + 65 = 3^4$$

$$5 = 1 + 2^2 = 1(1^2 + 2^2)$$
$$5 + 34 = 3 + 6^2 = (1 + 2)(2^2 + 3^2)$$
$$5 + 34 + 111 = 6 + 12^2 = (1 + 2 + 3)(3^2 + 4^2)$$

Let us focus on the fact that running total of the sums of the odd rows gives us a succession of fourth powers, or squares of squares. This is a generalization of the familiar fact that the sum of the first n odd numbers is the square of n. We can think of this as being implemented by writing the numbers $1, 2, \cdots$ in a column (or a succession of "rows" each with one number) and then taking a running total of the "sums" of the odd "rows".

Now let us move in the other direction:

Make a triangular array of the natural numbers with each row now containing *two* more numbers than the previous row.

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 2 & 3 & 4 & & \\
 & 5 & 6 & 7 & 8 & 9 & \\
10 & 11 & 12 & 13 & 14 & 15 & 16
\end{array}$$

The odd row sums of this array are 1, 35, 189, 559, $\cdots$ and their running totals are $1 = 1^2$, $36 = 2^2 \cdot 3^2 = (1+2+3)^2$, $225 = 3^2 \cdot 5^2 = (1+2+3+4+5)^2$, $784 = 4^2 \cdot 7^2 = (1+2+3+4+5+6+7)^2$, $\cdots$, You will notice that the square roots are the sum of odd sums of consecutive integers.Even sums are not to be left out. The even row sums in the array are 9, 91, 341, $\cdots$ and their running totals are $9 = 1^2 \cdot 3^2 = (1+2)^2$, $100 = 2^2 \cdot 5^2 = (1+2+3+4)^2$, $441 = 3^2 \cdot 7^2 = (1+2+3+4+5+6)^2$, $\cdots$.

We can continue in this vein. Writing the triangular array with 1, 4, 7, 10, $\cdots$ numbers in the consecutive rows, we find the running totals of the odd sums to be 1, $64 = 8^2 = 2^2 \cdot 4^2$, $441 = 21^2 = 3^2 \cdot 7^2$ and so on. With 1, 5, 9, 13, $\cdots$ numbers in the consecutive rows, the running totals of the odd sums are 1, $100 = 10^2 = 2^2 \cdot 5^2$, $729 = 27^2 = 3^2 \cdot 7^2$.

22. The diagonals of a concyclic quadrilateral $ABCD$ intersect in a point O. Establish the inequality

$$\frac{AB}{CD} + \frac{CD}{AB} + \frac{BC}{AD} + \frac{AD}{BC} \leq \frac{OA}{OC} + \frac{OC}{OA} + \frac{OB}{OD} + \frac{OD}{OB} .$$

23. Let A_1, A_2, $\cdots$, A_r be subsets of $\{1, 2, \cdots, n\}$ such that no A_i contains another. Suppose that A_i has a_i elements $(1 \leq i \leq r)$. Prove that

$$\sum \binom{n}{a_i}^{-1} \leq 1 .$$

24. Without recourse to a calculator or a computer, give an argument that $512^3 + 675^3 + 720^3$ is composite.

Solutions

Problem 19.

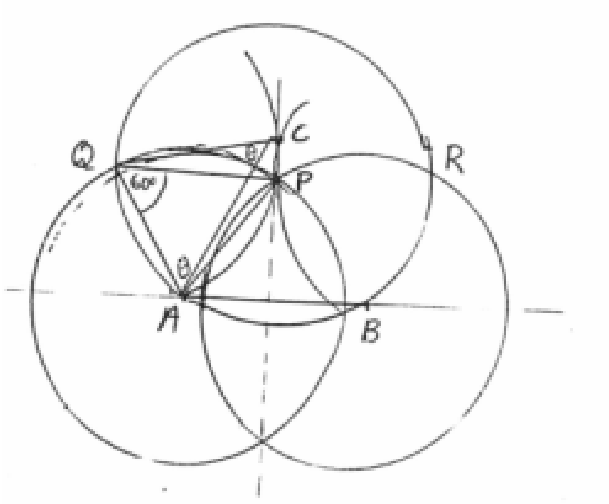
19. *First solution.* Method (i) leads to the equation $0 = (5u - 2v)(5u + 2v)$ while (ii) leads to $4u^2 = 9u^2 + v^2 - 9uv/2$ or $0 = (5u - 2v)(2u - v)$. Let us examine closely the second answer that is provided by (ii). In this case, the triangle formed by the two vectors and their resultant is isosceles with sides of magnitudes $3u$, $2u$ and $2u$ with the base angle equal to θ and the apex angle equal to $180° - 2\theta$. In this configuration, $\cos\theta$ is indeed $3/4$, but the apex angle is not 2θ as specified in the statement of the problem. Indeed, method (i) made use of the angle between $\mathbf{R}$ and $\mathbf{G}$. However in method (ii) the result of the sine law remained valid with $180° - 2\theta$ in place of 2θ but the cosine law did not make use of the angle between $\mathbf{R}$ and $\mathbf{G}$. So it is not surprising that the second method leads to a spurious possibility.

Problem 20.

20. There are various configurations, and we give the solution for one of these. The solutions for the other are similar.

20. *First solution.* `See Figure 20.1.` We have that $AP = AQ = QP = QC$. Let $\angle QAC = \angle QCA = \theta$, so that $\angle AQC = 180° - 2\theta$. Then $\angle PQC = \angle AQC - 60° = 120° - 2\theta$, so that $\angle QCP = \angle QPC = 30° + \theta$. Thus, $\angle ACP = 30°$. Since $CP \perp AB$, $\angle CAB = 60°$. Similarly, $\angle CBA = 60°$ and so ΔABC is equilateral.

FIGURE 20.1



20. *Second solution.* `See Figure 20.2.`Consider the reflection in the right bisector of AB. It interchanges circles with centres A and B, and hence fixes their intersection points, in particular P. This reflection also carries the circle with centre P to itself. Hence the point C is fixed by the reflection. It follows that P and C are on the right bisector of AB.

    Suppose that PC intersects AB in S. Let $\angle SAP = \theta$. Then $\angle APB = 180° - 2\theta$. Also

    $$\angle QPC = 180° - \angle APQ - \angle APS = 180° - 60° - (90° - \theta) = 30° + \theta$$

    whence

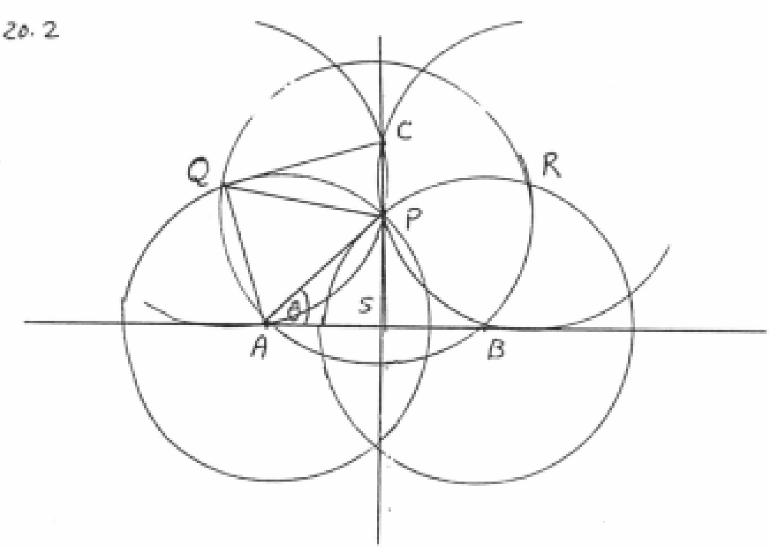
    $$\angle PQC = 180° - 2(30° + \theta) = 120° - 2\theta$$

    and

    $$\angle AQC = 60° + \angle PQC = 180° - 2\theta\ .$$

    Hence $\Delta AQC \equiv \Delta APB$ (SAS) so that $AC = AB$ and the result follows.

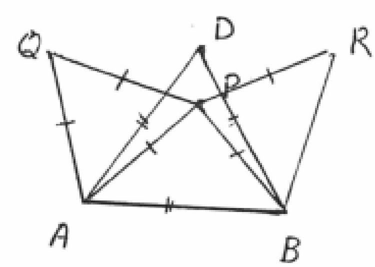
FIGURE 20.2



Third solution. `See Figure 20.3.` A $60°$ clockwise rotation about A followed by a $60°$ clockwise rotation about B takes $A \rightarrow A \rightarrow D$ and $Q \rightarrow P \rightarrow R$, where D is the third vertex of an equilateral triangle with side AB. Hence $AQ \rightarrow DR$ so that $|DR| = |AQ|$, the radius of the circle.

Similarly, the composition of two counterclockwise rotations with respective centres B and A takes $B \rightarrow B \rightarrow D$ and $R \rightarrow P \rightarrow Q$, so that $|DQ| = |BR|$, the radius of the circle. Hence D is a point of intersection of the circles with centres Q and R. When the radius is not equal to $|AB|$, this will be distinct from P, so that $D = C$, and the result follows.

FIGURE 20.3.



Problem 21.

21. *First solution.* Consider the triangular array in which the consecutive positive integers are written in rows with only the number 1 in the top row and with $k \geq 0$ more elements in each row than the previous one. For $r \geq 1$, the rth row has $(r-1)k+1$ elements beginning with $\binom{r-1}{2}k+r$ and ending with $\binom{r}{2}k+r$. The sum of the numbers in the rth row is

$$\frac{1}{2}[(r-1)k+1]\left[\left(\binom{r-1}{2}+\binom{r}{2}\right)k+2r\right]$$
$$= \frac{1}{2}[(r-1)k+1][(r-1)^2k+2r]$$
$$= \frac{1}{2}[(r-1)^3k^2+(3r^2-4r+1)k+2r] \ .$$

When $r = 2s-1$, this sum is

$$4(s-1)^3k^2+(6s^2-10s+4)k+(2s-1)$$

and the sum of the first m odd-numbered rows is the sum of these terms over $1 \leq s \leq m$, namely

$$[m(m-1)k]^2+2m^2(m-1)k+m^2 = [m((m-1)k+1)]^2 \ .$$

When $k = 0$, the sum of the elements in the sth odd-numbered row is $2s-1$ and the sum of the elments in the first m odd-numbered rows is m^2, the sum of the first m odd integers.

When $k = 1$, the sum of the elements in the sth odd-numbered row is

$$\begin{aligned} 4(s-1)^3 + (6s^2 - 10s + 4) + (2s-1) &= 4s^3 - 6s^2 + 4s - 1 \\ &= s^4 - (s-1)^4 = [s^2 - (s-1)^2][s^2 + (s-1)^2] \\ &= [(s-1) + s][(s-1)^2 + s^2] \end{aligned}$$

and the sum of all the elements in the first m odd-numbered rows is m^4.

When $k = 2$, the sum of the elements in the first m odd-numbered rows is

$$[m(2m-1)]^2 = [(1/2)(2m)(2m-1)]^2 = [1 + 2 + \cdots + \overline{2m-1}]^2 \quad .$$

When $r = 2s$, the sum of the elements in the rth row is

$$\frac{1}{2}[(2s-1)^3k^2 + (12s^2 - 8s + 1)k + 4s] \; .$$

For $k = 0$, this sum is $2s$. When $k = 1$, the sum is $4s^3 + s = s[1 + (2s)^2]$ and the sum of all the elements in the first m even-numbered rows is

$$\begin{aligned} m^2(m+1)^2 + \frac{1}{2}m(m+1) &= \binom{m+1}{2}[m^2 + (m+1)^2] \\ &= [1 + 2 + \cdots + m][m^2 + (m+1)^2] \quad . \end{aligned}$$

For $k = 2$, the sum of the numbers in the sth even-numbered row is

$$2(2s-1)^3 + (12s^2 - 8s + 1) + 2s = 16s^3 - 12s^2 + 6s - 1 = (2s-1)^3 + (2s)^3$$

and the sum of all the elements in the first m even-numbered rows is

$$\sum_{s=1}^{m}[(2s-1)^3 + (2s)^3] = \sum_{r=1}^{2m} r^3 = [1 + 2 + \cdots + 2m]^2 \; .$$

Problem 22.

22. *First solution.* Let the points and lengths be as labelled in the diagram, and let $\theta = \angle AOB$, $\alpha = \angle BAD$, $\beta = \angle ABC$. Then $\angle BCD = 180° - \alpha$ and $\angle CDA = 180° - \beta$. Then, where $[\cdots]$ denotes area,

$$2[ABC] = ab \sin \beta = (p+q) r \sin \theta$$

$$2[ACD] = cd \sin \beta = (p+q) s \sin \theta$$

so that $r/s = (ab)/(cd)$. Similarly $p/q = (ad)/(bc)$. Hence

$$\frac{p}{q} + \frac{r}{s} = \frac{a}{c}\left(\frac{d}{b} + \frac{b}{d}\right) \geq 2\frac{a}{c}$$

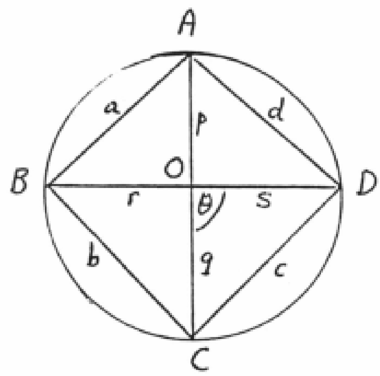
$$\frac{p}{q} + \frac{s}{r} = \frac{d}{b}\left(\frac{a}{c} + \frac{c}{a}\right) \geq 2\frac{d}{b}$$

$$\frac{q}{p} + \frac{r}{s} = \frac{b}{d}\left(\frac{c}{a} + \frac{a}{c}\right) \geq 2\frac{b}{d}$$

$$\frac{q}{p} + \frac{s}{r} = \frac{c}{a}\left(\frac{b}{d} + \frac{d}{b}\right) \geq 2\frac{c}{a}$$

and the result follows.

FIGURE 22.1



Second solution. From similar triangles, we find that $a/c = r/q = p/s$, $b/d = r/p = q/s$, so that $pq = rs$. Then

$$\begin{aligned}\left(\frac{p}{q} + \frac{q}{p} + \frac{r}{s} + \frac{s}{r}\right) &- \left(\frac{a}{c} + \frac{c}{a} + \frac{b}{d} + \frac{d}{b}\right) \\ &= \left(\frac{p}{q} + \frac{q}{p} + \frac{r}{s} + \frac{s}{r}\right) - \left(\frac{r}{q} + \frac{s}{p} + \frac{q}{s} + \frac{p}{r}\right) \\ &= \frac{p^2 + q^2 - pr - qs}{pq} + \frac{r^2 + s^2 - qr - ps}{rs} \ .\end{aligned}$$

Since $pq = rs$, this is a fraction over a common denominator with numerator

$$\begin{aligned}p^2 + q^2 + r^2 &+ s^2 - pr - qs - rq - sp \\ &= \frac{1}{2}[(p-r)^2 + (q-s)^2 + (q-r)^2 + (s-p)^2] \ ,\end{aligned}$$

which is nonnegative. The result follows.

Third solution. Begin as in the second solution. Then

$$\begin{aligned}\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{b}{b} &= \frac{r}{q} + \frac{q}{s} + \frac{s}{p} + \frac{p}{r} \\ &= \frac{pr + qs}{pq} + \frac{rq + sp}{rs} \\ &= \frac{pr + qs + rq + sp}{pq} \\ &\leq \frac{\sqrt{p^2 + q^2 + r^2 + s^2}\sqrt{r^2 + s^2 + q^2 + p^2}}{pq} \\ &= \frac{p^2 + q^2 + r^2 + s^2}{pq} \\ &= \frac{p^2 + q^2}{pq} + \frac{r^2 + s^2}{rs} \\ &= \frac{p}{q} + \frac{q}{p} + \frac{r}{s} + \frac{s}{r} \ ,\end{aligned}$$

from the Cauchy-Schwarz Inequality.

Problem 23.

23. *First solution.* There are $n!$ arrangements of the first n natural numbers. Suppose that $a_i = k$; consider the arrangements $\{x_1, x_2, \cdots, x_n\}$ where $A_i = \{x_1, x_2, \cdots, x_k\}$ (*i.e.*, the first a_i numbers constitute the set A_i). There are $k!$ possible ways of ordering the first k numbers and $(n-k)!$ ways of ordering the remaining numbers so that there are $a_i!(n-a_i)!$ arrangements of this type. Let $i \neq j$. No arrangement $\{x_1, x_2, \cdots, x_n\}$ can at the same time have its first a_i elements coincide with A_i and its first a_j elements coincide with a_j, since neither of A_i and A_j is contained in the other. Hence $a_i!(n-a_i)!$ arrangements corresponding to A_i are distinct from the $a_j!(n-a_j)!$ arrangements corresponding to A_j. It follows that

$$\sum_{i=1}^{n} a_i!(n-a_i)! \leq n!$$

and the result obtains.

Problem 24.

24. *First solution.* Observe that $512 = 2^9$, $675 = 3^3 \cdot 5^2$ and $720 = 2^4 \cdot 3^2 \cdot 5$, so that $2 \cdot 720^2 = 3 \cdot 512 \cdot 675$. We now use the identity

$$x^3 + y^3 + z^3 = x^3 + y^3 - z^3 + 3xyz = (x+y-z)(x^2 + y^2 + z^2 - xy + xz + yz)$$

which is valid when $2z^2 = 3xy$ to obtain that

$$512^3 + 675^3 + 720^3 = (512 + 675 - 720)(512^2 + 675^2 + 720^2 - 512 \cdot 675 + 512 \cdot 720 + 675 \cdot 720) \ .$$

Thus, $467 = 512 + 675 - 720$ is a factor of the sum of cubes.

24. *Second solution.* [P. LeVan]

$$\begin{aligned} 512^3 + 675^3 + 720^3 &= 512^3 + 45^3[15^3 + 16^3] \\ &= 512^3 + 45^3[(16-1)^3 + 16^3] \\ &= 512^3 - 45^3 + 45^3 \cdot 16[2 \cdot 16^2 - 3 \cdot 16 + 3] \\ &= (512 - 45)(512^2 + 512 \cdot 45 + 45^2) + 45^3 \cdot 16[512 - 3 \cdot 15] \\ &= 467[512^2 + 512 \cdot 45 + 45^2] + 45^3 \cdot 16[467] \\ &= 467[512^2 + 512 \cdot 45 + 45^2 + 45^3 \cdot 16] \end{aligned}$$

so that 467 is a factor of the sum of the cubes.

24. *Third solution.* [D. Pritchard] We have the identity

$$(a^9)^3 + (b^3c^2)^3 + (a^4b^2c^3)^3 = a^{27} + b^9c^6 + a^{12}b^6c^3$$

$$= (a^9 + b^3c^2 - a^4b^2c)(a^{18} - a^9b^3c^2 + b^6c^4 + a^{13}b^2c + a^4b^5c^3 + a^8b^4c^2) + (2b - 3a)a^{12}b^5c^3 \ .$$

Now take $(a, b, c) = (2, 3, 5)$ to yield the desired result, one factor being 467.

24. *Fourth solution.* [D. Arthur] Writing $512 = \frac{1}{2}(x+y)$, $675 = \frac{1}{2}(x+z)$ and $720 = \frac{1}{2}(y+z)$ yields $(x, y, z) = (467, 557, 883)$. Modulo 467, we find that

$$\begin{aligned} 512^3 + 675^3 + 720^3 &\equiv 8^{-1}(2y^3 + 2z^3 + 3y^2z + 3yz^2) \\ &= 8^{-1}(y+z)(2y^2 + 2z^2 + yz) \ . \end{aligned}$$

[Note that 8^{-1} represents a number a for which $8a \equiv 1 \pmod{467}$.] Now $y \equiv 90$ and $z \equiv -51$, so that

$$\begin{aligned} 2y^2 + 2z^2 + yz &\equiv 2 \times 90^2 + 2 \times 51^2 - 90 \times 51 \\ &= 9[4 \times 450 + 2 \times 289 - 510] \\ &\equiv 9[-4 \times 17 + 578 - 510] = 9[-68 + 68] = 0 \end{aligned}$$

and so the sum of the cubes is divisible by 467.

24. *Comment.* In fact,

$$512^3 + 675^3 + 720^3 = 229 \times 467 \times 7621 \ .$$