(solutions follow)

1998-1999 Olympiad Correspondence Problems

Set 5

25. Let c be a positive integer and let $f_0 = 1$, $f_1 = c$ and $f_n = 2f_{n-1} - f_{n-2} + 2$ for $n \geq 1$. Prove that for each $k \geq 0$, there exists $m \geq 0$ for which $f_k f_{k+1} = f_m$.

26. Let $p(x)$ be a polynomial with integer coefficients for which $p(0) = p(1) = 1$. Let a_0 be any nonzero integer and define $a_{n+1} = p(a_n)$ for $n \geq 0$. Prove that, for distinct nonnegative integers i and j, the greatest common divisor of a_i and a_j is equal to 1.

27. (a) Let n be a positive integer and let $f(x)$ be a quadratic polynomial with real coefficients and real roots that differ by at least n. Prove that the polynomial $g(x) = f(x)+f(x+1)+f(x+2)+\cdots+f(x+n)$ also has real roots.

    (b) Suppose that the hypothesis in (a) is replaced by positing that $f(x)$ is any polynomial for which the difference between any pair of roots is at least n. Does the result still hold?

28. Find the locus of points P lying inside an equilateral triangle for which $\angle PAB + \angle PBC + \angle PCA = 90°$.

29. (a) Prove that for an arbitrary plane convex quadrilateral, the ratio of the largest to smallest distance between pairs of points is at least $\sqrt{2}$.

    (b) Prove that for any six distinct points in the plane, the ratio of the largest to smallest distances between pairs of them is at least $\sqrt{3}$.

30. Let a, b, c be positive real numbers. Prove that

    (a) $4(a^3 + b^3) \geq (a + b)^3$;

    (b) $9(a^3 + b^3 + c^3) \geq (a + b + c)^3$,

    (c) More generally, establish that for each integer n and n nonnegative reals that

$$n^2(a_1^3 + a_2^3 + \cdots + a_n^3) \geq (a_1 + a_2 + \cdots + a_n)^3 \quad .$$

Solutions

25. *First solution.* A straightforward induction argument establishes that $f_n = nc + (n-1)^2$, and it can be checked algebraically that

$$f_k f_{k+1} = f_m$$

where $m = kc + (k^2 - k + 1) = f_k + k$.

25. *Comment.* This is basically a problem in experimentation and pattern recognition. Once the result is conjectured, the algebraic manipulations required are straightforward. One way to approach the problem is to consider $f_k f_{k+1} = m^2 + (c-2)m + 1$ as a quadratic equation to be solved for m. Note that when $c = 4$, it turns out that $f_n = (n+1)^2$. If we let $f(c, n) = nc + (n-1)^2$, then $m = f(c+1, k)$. More generally, we have for every monic quadratic polynomial p with integer coefficients, $p(k)p(k+1) = p(m)$ for some integer m.

Problem 26.

26. *First solution.* By the factor theorem, we have that $p(x) - 1 = x(x-1)q(x)$ for some polynomial $q(x)$ with integer coefficients. Hence

$$\begin{aligned} a_{n+1} - 1 &= a_n(a_n - 1)q(a_n) \\ &= \cdots = a_n a_{n-1} \cdots a_1(a_1 - 1) \prod_{k=1}^{n} q(a_k) \end{aligned}$$

for each positive integer n. (Note that a_{n+1} cannot be zero, as this would force the impossible $a_n(a_n - 1)q(a_n) = -1$.) It follows that $a_m \equiv 1 \pmod{a_r}$ whenever $m > r$ and the result follows.

26. *Second solution.* Let $p(x) = c_d x^d + \cdots + c_1 x + c_0$. The conditions imply that $c_0 = c_d + \cdots + c_1 + c_0 = 1$. Let a be any nonzero integer. Then

$$p(a) = c_d a^d + \cdots + c_1 a + c_0 \equiv c_0 = 1 \pmod{a} .$$

Suppose as an induction hypothesis that the ith iterate $p^i(a) \equiv 1 \pmod{a}$. Then

$$p^{i+1}(a) = c_d p^i(a)^d + \cdots + c_1 p^i(a) + c_0 \equiv c_d + \cdots + c_1 + c_0 = 1 \pmod{a} .$$

In particular, we find that $a_{n+r} \equiv 1 \pmod{a_n}$ for each pair of positive integers n and r from which the result follows.

Problem 27.

27. (a) *First solution.* By making a substitution of the form $x' = x - k$, we can dispose of the linear term in the quadratic. Thus we may assume that $f(x) = x^2 - r^2$ where $2r \geq n$. Then

$$\begin{aligned} g(x) &= (n+1)[x^2 - nx + \frac{n(2n+1)}{6} - r^2] \\ &= (n+1)\left[\left(x + \frac{n}{2}\right)^2 + \left(\frac{n^2 + 2n}{12} - r^2\right)\right] . \end{aligned}$$

Since

$$r^2 - \frac{n^2 + 2n}{12} \geq \frac{n^2}{4} - \frac{n^2 + 2n}{12} = \frac{n(n-1)}{6} ,$$

$g(x) \leq 0$ when $x = -n/2$ and so has real roots. These roots are coincident if and only if $n = 1$ and $r = 1/2$, *i.e.*, when $f(x) = x^2 - \frac{1}{4}$.

27. (a) *Second solution.* [Z. Amir-Khosravi] Let u and v be the two roots of $f(x)$ with $u + n \leq v$, with strict inequality when $n = 1$. Wolog, suppose that $f(x) < 0$ for $x < u$ and $x > v$, while $f(x) < 0$ for $u < x < v$. Then $g(x)$ is a quadratic polynomial for which

$$\begin{aligned} g(u-n) &= f(u-n) + \cdots + f(u) > 0 \\ g(u) &= f(u) + \cdots + f(u+n) < 0 \\ g(v) &= g(v) + \cdots + g(v+n) > 0 . \end{aligned}$$

By the Intermediate Value Theorem, $g(x)$ has two roots.

When $n = 1$ and $v = u + 1$, then $g(u) = 0$. Since $g(x)$ is a real quadratic with at least one real root, both roots are real.

27. (b) *First solution.* Suppose that $f(x)$, and hence $g(x)$, has degree d, and let the roots of $f(x)$ be $a_1, a_2, \cdots, a_d$ where $a_{i+1} \geq a_i + n$. Since $f(x)$ has no multiple roots, it alternates in sign as x passes through the roots a_i.

To begin with, let $n \geq 2$, and let a_i be one of the roots. Suppose, wolog, that $f(x) < 0$ for $a_i - n < x < a_i$ and $f(x) > 0$ for $a_i < x < a_i + n$. Then

$$g(a_i - n) = f(a_i - n) + \cdots + f(a_i - 1) + f(a_i) < 0$$

and

$$g(a_i) = f(a_i) + f(a_i + 1) + \cdots + f(a_i + n) > 0 \ .$$

By the Intermediate Value Theorem, $g(x)$ has a root b_i between $a_i - n$ and a_i. Since the open intervals $(a_i - n)$ are disjoint, the b_i are distinct. Since we have identified d (the degree of $g(x)$) roots of $g(x)$, all of the roots of $g(x)$ are real.

Now suppose that $n = 1$. If the intervals $[a_i - 1, a_i]$ are all disjoint, then we can use the foregoing argument since $g(a_i - 1) = f(a_i - 1)$ and $g(a_i) = f(a_i + 1)$ are nonzero with opposite signs. Suppose, on the other hand, there are s roots of $f(x)$, namely $a_r, a_{r+1}, \cdots, a_{r+s-1}$ spaced unit distance apart ($a_{r+i} = a_{r+i-1} + 1$ for $1 \leq i \leq s-1$) such that $a_r - 1$ and $a_{r+s-1} + 1$ are not roots of $f(x)$. We will discuss the case that $f(x) < 0$ for $a_{r-1} \leq x < a_r$ and $f(x) > 0$ for $a_{r+s-1} < x \leq a_{r+s-1} + 1$; the other three cases can be handled analogously.

In this case, all the roots of $f(x)$ are simple, s is odd, and

$$g(a_r - 1) = f(a_r - 1) + f(a_r) = f(a_r - 1) < 0$$

$$g(a_{r+i}) = f(a_{r+i}) + f(a_{r+i+1}) = 0 \qquad \text{for} \quad 0 \leq i \leq s-2$$

and

$$g(a_{r+s-1}) = f(a_{r+s-1}) + f(a_{r+s-1} + 1) = f(a_{r+s-1} + 1) > 0 \ .$$

Thus, $g(x)$ must have an odd number of roots in the interval $(a_r - 1, a_{r+s-1})$ counting multiplicity. We can account for $s-1$ of them, $\{a_r, a_{r+1}, \cdots, a_{r+s-2}\}$, so, since $s-1$ is even, there must be at least s roots in this interval.

We can apply this analysis to the other "blocks" of s roots of $f(x)$ spaced at unit distance to obtain disjoint intervals with at least s roots of $g(x)$. From this, we can deduce that all the roots of $g(x)$ must be real.

Problem 28.

28. *First solution.* Let $\angle PAB = \alpha$, $\angle PAC = \alpha'$, $\angle PBC = \beta$, $\angle PBA = \beta'$, $\angle PCA = \gamma$ and $\angle PCB = \gamma'$, so that $\alpha + \alpha' = \beta + \beta' = \gamma + \gamma' = 60°$. Let $|AP| = u$, $|BP| = v$ and $|CP| = w$. By the Law of Sines,

$$\frac{\sin\alpha}{\sin\beta'} \cdot \frac{\sin\beta}{\sin\gamma'} \cdot \frac{\sin\gamma}{\sin\alpha'} = \frac{v}{u} \cdot \frac{w}{v} \cdot \frac{u}{w} = 1$$

whence

$$\sin\alpha \sin\beta \sin\gamma = \sin\alpha' \sin\beta' \sin\gamma' \ .$$

Now

$$\begin{aligned}\sin\alpha\sin\beta\sin\gamma &= \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]\sin\gamma \\ &= \frac{1}{4}[\sin(\gamma+\alpha-\beta) + \sin(\beta+\gamma-\alpha) + \sin(\alpha+\beta-\gamma) - \sin(\alpha+\beta+\gamma)] \ .\end{aligned}$$

Suppose that $\alpha + \beta + \gamma = 90°$. Then

$$\begin{aligned}\sin\alpha\sin\beta\sin\gamma &= \frac{1}{4}[\sin(90° - 2\alpha) + \sin(90° - 2\beta) + \sin(90° - 2\gamma) - 1] \\ &= \frac{1}{4}[\cos 2\alpha + \cos 2\beta + \cos 2\gamma - 1] \ .\end{aligned}$$

Since also $\alpha' + \beta' + \gamma' = 90°$, a similar identity holds for α', β' and γ'. Hence

$$\begin{aligned} 0 &= (\cos 2\alpha - \cos 2\alpha') + (\cos 2\beta - \cos 2\beta') + (\cos 2\gamma - \cos 2\gamma') \\ &= 2\sin 60°[\sin\lambda + \sin\mu + \sin\nu] \end{aligned}$$

where $\lambda = \alpha - \alpha'$, $\mu = \beta - \beta'$ and $\nu = \gamma - \gamma'$. Note that $|\lambda|, |\mu|, |\nu| < 60°$ and that $\lambda + \mu + \nu = 0$.

Now $\sin\lambda + \sin\mu = 2\sin\frac{1}{2}(\lambda+\mu)\cos\frac{1}{2}(\lambda-\mu) = -2\sin\frac{\nu}{2}\cos\frac{\lambda-\mu}{2}$ and $\sin\nu = 2\sin\frac{\nu}{2}\cos\frac{\nu}{2}$, whence $0 = 2\sin\frac{\nu}{2}[\cos\frac{\nu}{2} - \cos\frac{\lambda-\mu}{2}]$. Hence $\nu = 0$, $\nu = \lambda - \mu$ or $\nu = \mu - \lambda$. If $\nu = 0$, then $\gamma = \gamma'$ and P lies on the bisector of angle C. If $\lambda = \mu + \nu$, then $\lambda = 0$ and P lies on the bisector of angle A. If $\mu = \lambda + \nu$, then $\mu = 0$ and P must lie on the angle bisector of the triangle ABC.

Conversely, it is easily checked that any point P on these bisectors satisfies the given condition.

28. *Second solution.* [Z. Amir-Khosravi] `See Figure 28.2.` Let the vertices of the triangle in the complex plane be 1, -1 and $\sqrt{3}i$, and let z be a point on the locus. Then, with the angles as indicated in the diagram and $\text{cis}\ \theta = \cos\theta + i\sin\theta$, we have that

$$\frac{1-\sqrt{3}i}{2} = \frac{z-\sqrt{3}i}{|z-\sqrt{3}i|}\text{cis}\ \alpha$$

$$1 = \frac{1-z}{|1-z|}\text{cis}\ \beta$$

and

$$\frac{\sqrt{3}i+1}{2} = \frac{z+1}{|z+1|}\text{cis}\ \gamma\ .$$

Since $\text{cis}\ \alpha \cdot \text{cis}\ \beta \cdot \text{cis}\ \gamma = \text{cis}\ \frac{\pi}{2} = i$,

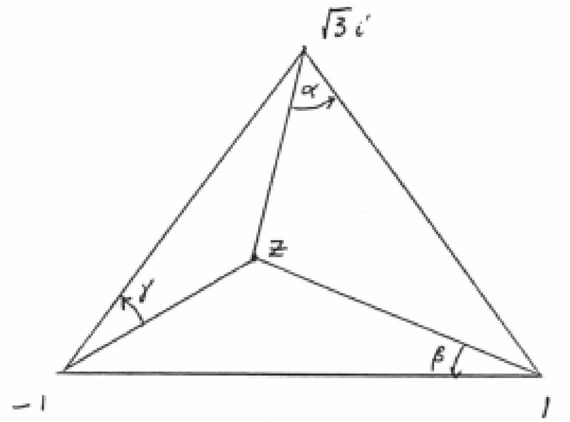
$$1 = i\frac{(z-\sqrt{3}i)(1-z^2)}{|z-\sqrt{3}i||1-z^2|}\ .$$

Hence the real part of $(z-\sqrt{3}i)(1-z^2)$ is 0, which implies that the real part of $z - z^3 + \sqrt{3}iz^2$ is 0. Setting $z = x + yi$, we find that

$$\begin{aligned} 0 &= x - x^3 + 3xy^2 - 2\sqrt{3}xy \\ &= x[(1-\sqrt{3}y)^2 - x^2] \\ &= x(1-\sqrt{3}y+x)(1-\sqrt{3}y-x)\ . \end{aligned}$$

Hence (x, y) lies on one of the lines $x = 0$, $y = (1/\sqrt{3})(1+x)$, $y = (1/\sqrt{3})(1-x)$, the three bisectors of the angles. Conversely, one can show that any point on the bisectors lies on the lines.

FIGURE 28.2



Problem 29.

29. *First solution.* (a) One of the angles θ of the quadrilateral must be between $90°$ and $180°$ inclusive so that $\cos\theta \leq 0$. Let a and b be the lengths of the sides bounding this angle and let u be the length of the diagonal joining the endpoints of these sides (opposite θ). Wolog, suppose that $a \leq b$. Then

$$u^2 = a^2 + b^2 - 2ab\cos\theta \geq a^2 + b^2 \geq 2a^2$$

whence $u^2/a^2 \geq 2$. Since the largest distance is at least u and the smallest at most a, the result follows.

(b) If ABC is a triangle for which $a \leq b \leq c$ and $C \geq 120°$, then $-1 \leq \cos C \leq -1/2$ and

$$c^2 = a^2 + b^2 - 2ab\cos 120° \geq a^2 + b^2 + ab \geq 3a^2$$

so that $c \geq \sqrt{3}a$. Thus to prove the result, we need to show that among the six given points, we can find three that are vertices of a (possibly degenerate with three vertices in a line) triangle with one angle at least $120°$, or in other words, a point rays from which to two other points make an angle of at least $120°$.

Consider the smallest closed convex set containing the six points. If all six points are on the boundary, then they are the vertices of a (possibly degenerate) convex hexagon. Since the average interior angle of such a hexagon is $120°$, one of the angles is at least $120°$ and we get the desired result.

If the convex set has fewer than six points on the boundary, the remaining point(s) lie in its interior. Triangulate the convex set using only the boundary points; one of the interior points lies inside one of the triangles (possibly on a side), and one of the sides of the triangle must subtend at this point an angle of at least $120°$. Again, the desired result holds.

29. *Second solution.* [L. Lessard] Suppose ABC is a triangle with $a \leq b \leq c$ and $90° \leq C \leq 180°$. Then $2A \leq A + B = 180° - C$, so that by the Law of Sines,

$$\frac{c}{a} = \frac{\sin C}{\sin A} \geq \frac{\sin C}{\sin(90° - C/2)} = \frac{2\sin(C/2)\cos(C/2)}{\cos(C/2)} = 2\sin(C/2)\ .$$

If $C = 90°$, then $c \geq \sqrt{2}a$ and if $C \geq 120°$, then $c \geq \sqrt{3}a$. The solution can be completed as before.

Problem 30

The specific results

30. *First solution.* (a) Taking the difference of the two sides of the inequality yields 3 times

$$a^3 - a^2b - ab^2 + b^3 = (a^2 - b^2)(a + b) = (a + b)(a - b)^2 \geq 0 \ .$$

(b) Using the result in (a), we obtain that

$$\begin{aligned} 9(a^3 + b^3 + c^3) &= 4(a^3 + b^3) + 4(a^3 + c^3) + 4(b^3 + c^3) + (a^3 + b^3 + c^3) \\ &\geq (a + b)^3 + (a + c)^3 + (b + c)^3 + (a^3 + b^3 + c^3) \\ &= (a + b + c)^3 + 2(a^3 + b^3 + c^3) - 6abc \\ &\geq (a + b + c)^3 + 2(a^3 + b^3 + c^3) - 2(a^3 + b^3 + c^3) \\ &= (a + b + c)^3 \ , \end{aligned}$$

with the last inequality due to that of the arithmetic and geometric means.

30. *Second solution.* (b) Since the desired inequality is completely symmetrical in the variables, wolog we may suppose that $a \geq b \geq c$. Then

$$a^2(a - b) + b^2(b - c) + c^2(c - a) = (a^2 - c^2)(a - b) + (b^2 - c^2)(b - c) \geq 0$$

and

$$a^2(a - c) + b^2(b - a) + c^2(c - b) = (a^2 - b^2)(a - c) + (b^2 - c^2)(b - c) \geq 0 \ ,$$

so that

$$a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$$

and

$$a^3 + b^3 + c^3 \geq ab^2 + bc^2 + ca^2 \ .$$

Hence

$$\begin{aligned} (a + b + c)^3 &= a^3 + b^3 + c^3 + 3(a^2b + b^2c + c^2a) + 3(ab^2 + bc^2 + ca^2) + 6abc \\ &\leq a^3 + b^3 + c^3 + 3(a^3 + b^3 + c^3) + 3(a^3 + b^3 + c^3) + 2(a^3 + b^3 + c^3) \end{aligned}$$

from which the result follows.

The general result

30. *First solution.* The Chebyshev Inequality asserts that if $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$ and $y_1 \geq y_2 \geq \cdots \geq y_n \geq 0$, then

$$n(x_1y_1 + x_2y_2 + \cdots + x_ny_n) \geq (x_1 + \cdots + x_n)(y_1 + \cdots + y_n) \ ,$$

This can be seen by taking the difference of the two sides which turns out to be the sum of all nonnegative products of the form $(x_i - x_j)(y_i - y_j)$ with $i < j$.

Using this twice yields

$$\begin{aligned} n^2(a_1^3 + \cdots + a_n^3) &\geq n(a_1^2 + \cdots + a_n^2)(a_1 + \cdots + a_n) \\ &\geq (a_1 + \cdots + a_n)^2(a_1 + \cdots + a_n) = (a_1 + \cdots + a_n)^3 \ , \end{aligned}$$

30. *Second solution.* The power mean inequality asserts that, for $k > 1$,

$$\frac{a_1 + \cdots + a_n}{n} \le \left(\frac{a_1^k + \cdots + a_n^k}{n}\right)^{1/n} .$$

Applying this to $k = 3$ yields the result.

30. *Third solution.* By the Cauchy-Schwarz Inequality, we obtain

$$\begin{aligned} n^2\left(\sum_{i=1}^n a_i^3\right)\left(\sum_{i=0}^n a_i\right) \ge n^2\left(\sum_{i=0}^n a_i^2\right)^{1/2} \\ \left[\left(\sum_{i=0}^n 1\right) \cdot \left(\sum_{i=0}^n a_i^2\right)\right]^2 \\ \ge \left[\sum_{i=1}^n a_i\right]^4 . \end{aligned}$$

The result follows.

30. *Comment.* L. Lessard began his solution by in effect replacing the original set $\{a_i\}$ be a set whose elemenets are a little closer to the mean of these numbers.The result holds for $n = 1$. We assume as an induction hypothesis that it holds for $n = K$, *viz.* that

$$k^2(a_1^3 + a_2^3 + \cdots + a_k^3) \ge (a_1 + a_2 + \cdots + a_k)^3$$

for all suitable a_i. Apply this to each $k-$tuple among $\{a_1, a_2, \cdots, a_{k+1}\}$ and add the results to obtain

$$k^3(a_1^3 + a_2^3 + \cdots + a_{k+1}^3) \ge (S - a_1)^3 + (S - a_2)^3 + \cdots + (S - a_{k+1})^3$$

where $S = a_1 + a_2 + \cdots + a_{k+1}$. Let $b_i = (S - a_i)/k$. Then $S = b_1 + b_2 + \cdots + b_k$ and so

$$a_1^3 + a_2^3 + \cdots + a_{k+1}^3 \ge b_1^3 + b_2^3 + \cdots + b_{k+1}^3 .$$

This process can be iterated. The idea is that the right side tends under the iterations to a $(k+1)$-fold sum each terms of which is $S/(k+1)$. But this requires an additional step to show that the maximum of $|S - a_i|$ does indeed decrease sufficiently as to tend to zero as we keep iterating.