Crux Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

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EUREKA

Vol. 2, No. 7

August-September 1976

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
A Chapter of the Ontario Association for Mathematics Education

Publié par le Collège Algonquin

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All communications about the content of the magazine (articles, problems, solutions, etc.) should be sent to the editor: Léo Sauvé, Mathematics Department, Algonquin College, 281 *Echo Drive, Ottawa, Ont., K1S* 1*N*4.

All changes of address and inquiries about subscriptions and back issues should be sent to the Secretary-Treasurer of COMA: F.G.B. Maskell, Algonquin College, 200 *Lees Ave., Ottawa, Ont., K1S* 0*C*5.

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CONTENTS

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COMING: A GAUSS BICENTENNIAL ISSUE

The April 1977 issue of EUREKA will be a Gauss bicentennial issue, to commemorate the birth of Karl Friedrich Gauss on April 30, 1777. Articles and problems relating to Gauss are earnestly solicited for this issue.

An Ottawa Gauss Bicentennial Committee has been formed to organize the celebrations for this event. These will include a day-long Symposium to be held in Ottawa on April 30, 1977, an exhibition of Gauss memorabilia, and an essay-writing Gauss Competition in which Ottawa area high school students will be invited to participate and for which prizes will be awarded.

Further details will appear in later issues of EUREKA.

* * *

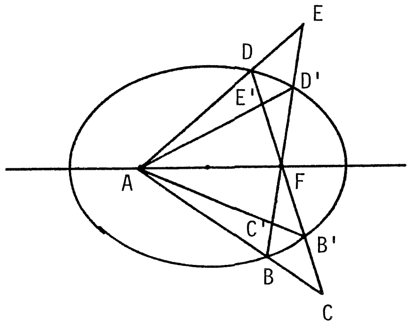
URQUHART'S THEOREM AND THE ELLIPSE

DAN EUSTICE, The Ohio State University

Let A and F be the foci of an ellipse with eccentricity e. If P is a point on the ellipse and a and b are the focal angles to the point P, then we find from the law of sines that $\sin(a+b) = e(\sin a + \sin b)$. Using half-angle and sum formulas, this becomes

$$\tan\frac{a}{2}\tan\frac{b}{2} = \frac{1-e}{1+e}. \tag{1}$$

From the reversibility of the calculation, we see that (1) is a characterization of the ellipse.



Choose points B and D on the ellipse (see figure) and let B' and D' be determined on the ellipse by the lines through DF and BF. Also let E, C, E', and C' be determined by the intersections of the appropriate lines. To avoid cluttering up the figure, we leave it to the reader to inscribe the following angles in a separate figure:

θ = DAF	α = D'AF	ω = BAF	γ = B'AF
ϕ = DFA	β = D'FA	η = BFA	δ = B'FA

From (1), we obtain

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \tan\frac{\theta}{2}\tan\frac{\phi}{2} = \tan\frac{\omega}{2}\tan\frac{\eta}{2} = \tan\frac{\gamma}{2}\tan\frac{\delta}{2} = \frac{1-e}{1+e}. \tag{2}$$

From the equality of the second and third terms of (2), we find, using the supplementary angles, that $\tan\frac{\theta}{2}\cot\frac{\delta}{2} = \tan\frac{\omega}{2}\cot\frac{\beta}{2}$ or

$$\tan\frac{\theta}{2}\tan\frac{\beta}{2} = \tan\frac{\omega}{2}\tan\frac{\delta}{2}. \tag{3}$$

But this implies that the points E and C are on an ellipse (of eccentricity, say, e_1) which is confocal with the given ellipse, and so AC + CF = AE + EF. This is the content of Urquhart's Theorem that AB + BF = AD + DF implies AC + CF = AE + EF.

If we use the first and fourth terms of (2), we obtain $\tan\frac{\alpha}{2}\cot\frac{\eta}{2} = \tan\frac{\gamma}{2}\cot\frac{\phi}{2}$, whence

$$\tan\frac{\alpha}{2}\tan\frac{\phi}{2} = \tan\frac{\gamma}{2}\tan\frac{\eta}{2}. \tag{4}$$

Thus E' and C' are also points on an ellipse (of eccentricity, say, e_2) which is confocal with the given one, and AE' + E'F = AC' + C'F.

From (2), (3), and (4), the eccentricities of the two confocal ellipses are related by

$$\frac{1-e_1}{1+e_1}\cdot\frac{1-e_2}{1+e_2} = \left(\frac{1-e}{1+e}\right)^2.$$

A calculation similar to that in the first paragraph, starting with

$$\sin(a+b) = e(\sin a - \sin b),$$

yields a characterization for hyperbolas with foci at A and F and eccentricity e'. We find

$$\tan\frac{a}{2}\cot\frac{b}{2} = \frac{e'+1}{e'-1}.$$

If B and D are on the same properly chosen branch of such a hyperbola and the remaining notation is the same, then the equivalent of (2) is

$$\tan\frac{\beta}{2}\cot\frac{\alpha}{2} = \tan\frac{\phi}{2}\cot\frac{\theta}{2} = \tan\frac{\eta}{2}\cot\frac{\omega}{2} = \tan\frac{\delta}{2}\cot\frac{\gamma}{2} = \frac{e'+1}{e'-1}, \tag{5}$$

and it is then easy to show that E and C (also E' and C') are on a confocal hyperbola. Thus $AB - BF = AD - DF$ implies $AC - CF = AE - EF$ and $AC' - C'F = AE' - E'F$.

Other implications of this type are easily deduced. For example, we get from (2)

$$\tan\frac{\alpha}{2}\cot\frac{\phi}{2} = \tan\frac{\theta}{2}\cot\frac{\beta}{2},$$

which shows that E and E' (in the original figure) are on a hyperbola with the same foci as the ellipse, and so $AB + BF = AD + DF$ implies $AE - EF = AE' - E'F$. Similarly from (5),

$$\tan\frac{\eta}{2}\tan\frac{\gamma}{2} = \tan\frac{\delta}{2}\tan\frac{\omega}{2},$$

which shows that C and C' (in the corresponding hyperbola figure) are on an ellipse with the same foci as the hyperbola, and so $AB - BF = AD - DF$ implies $AC + CF = AC' + C'F$.

REFERENCE

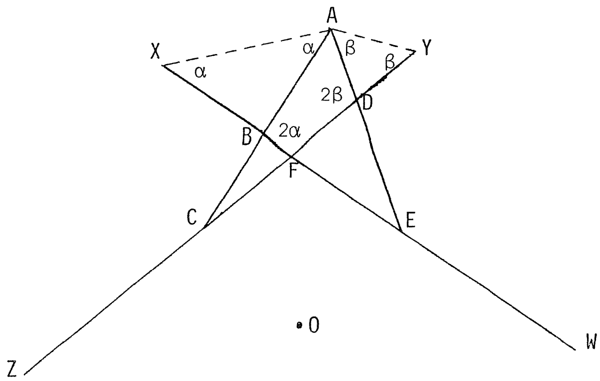
Léo Sauvé, On Circumscribable Quadrilaterals, *Eureka*, Vol. 2 (1976), pp. 63-67.

* * *

A "NO-CIRCLE" PROOF OF URQUHART'S THEOREM

DAN SOKOLOWSKY, Antioch College

This note is in response to Pedoe's invitation in [1,2] to find a proof of Urquhart's Theorem which does not involve circles. The proof given below uses only elementary geometry. See [3] for a trigonometric approach. The theorem itself is simply stated (see figure): $AB + BF = AD + DF$ implies $AC + CF = AE + EF$.



Proof. Let $BX = AB$, $DY = AD$, $CZ = AC$, and $EW = AE$; then

$$FX = AB + BF = AD + DF = FY,$$
$$FW = AE + EF,$$
$$FZ = AC + CF,$$

and it suffices to show that $FW = FZ$.

Let L_1 and L_2 denote the bisectors of angles ABX and ADY, respectively. Since Δs ABX and ADY are isosceles, L_1 and L_2 are right bisectors of AX and AY. Since $2\alpha + A < 180°$ in ΔABE and $2\beta + A < 180°$ in ΔACD, it follows that $\alpha + \beta + A < 180°$, and hence L_1 and L_2 intersect, say at O as in the figure, so that $OX = OA = OY$. Since $FX = FY$, Δs OFX and OFY are congruent and OF bisects $\angle WFZ$. Thus O is equidistant from WX and YZ. But O is equidistant from AB and WX, as well as from AD and YZ. Hence O is equidistant from all four lines AB, AD, WX, YZ. Hence OC bisects $\angle ACZ$; and since ΔACZ is isosceles, we have $OA = OZ$. Similarly OC bisects $\angle AEW$ and $OA = OW$. Therefore

$$OA = OW = OX = OY = OZ.$$

Additionally, the two isosceles Δs OWX and OYZ have equal altitudes from O; hence they are congruent and $WX = YZ$. Since $FX = FY$, we have

$$FW = WX - FX = YZ - FY = FZ.$$

REFERENCES

1. Dan Pedoe, The most elementary theorem of Euclidean geometry, *Mathematics Magazine*, Vol. 49 (1976), pp. 40-42.

2. Léo Sauvé, On circumscribable quadrilaterals, *Eureka*, Vol. 2 (1976), pp. 63-67.

3. Kenneth S. Williams, On Urquhart's elementary theorem of Euclidean geometry, *Eureka*, Vol. 2 (1976), pp. 108-109.

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LETTER TO THE EDITOR

Dear editor:

...I have just gotten around to reading the solution of Problem 110 [1976: 85-87] and I think it is a gem! My own inclinations make me prefer such a direct geometric method rather than trigonometric or analytic ones. The latter are just too strong and make me feel I am flying over beautiful terrain without seeing anything much...just "getting there". It is nice to know, as your solution shows, that this difficult problem can be met and dealt with successfully on its own terms...

EUREKA's "Letters to the Editor" department is not only enjoyable but provides an important forum for the exchange and cross-fertilization of ideas. I have found it stimulating and am sure other readers have as well.

DAN SOKOLOWSKY,
Yellow Springs, Ohio.

PROBLEMS--PROBLÈMES

Problem proposals, preferably accompanied by a solution, should be sent to the editor, whose name appears on page 131.

For the problems given below, solutions, if available, will appear in EUREKA *Vol.* 2, *No.* 10, *to be published around Dec.* 15, 1976. *To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than Dec.* 1, 1976.

161. *Proposed by Viktors Linis, University of Ottawa.*

Evaluate

$$\int_0^{\pi/2} \frac{\sin^{25} t}{\cos^{25} t + \sin^{25} t}\, dt .$$

162. *Proposed by Viktors Linis, University of Ottawa.*

If $x_0 = 5$ and $x_{n+1} = x_n + \dfrac{1}{x_n}$, show that

$$45 < x_{1000} < 45.1 .$$

This problem is taken from the list submitted for the 1975 Canadian Mathematics Olympiad (but not used on the actual exam).

163. *Proposed by Charles Stimler, Douglaston, N.Y.*

Find the value of the following infinite continued fraction:

$$\cfrac{2}{1+\cfrac{3}{2+\cfrac{4}{3+\cfrac{5}{4+\cfrac{6}{5+\ddots}}}}}$$

164. *Proposed by Steven R. Conrad, Benjamin N. Cardozo High School, Bayside, N.Y.*

In the five-digit decimal numeral $ABCDE$ $(A \neq 0)$, different letters do not necessarily represent different digits. If this numeral is the fourth power of an integer, and if $A + C + E = B + D$, find the digit C.

(This problem was originally written for the Fall 1975 Contest of the New York City Senior Interscholastic Mathematics League.)

165. *Proposed by Dan Eustice, The Ohio State University.*

Prove that, for each choice of n points in the plane (at least two distinct), there exists a point on the unit circle such that the product of the distances from the point to the chosen points is greater than one.

166. *Proposed by Steven R. Conrad, Benjamin N. Cardozo High School, Bayside, N.Y.*

Find a simple proof for the following problem, which is not new:

Prove that for all real x and positive integers k

$$\sum_{i=0}^{k-1}\left[x+\frac{i}{k}\right] = [kx],$$

where brackets denote the greatest integer function.

167. *Proposed by Léo Sauvé, Algonquin College.*

The first half of the Snellius-Huygens double inequality

$$\frac{1}{3}(2\sin\alpha+\tan\alpha) > \alpha > \frac{3\sin\alpha}{2+\cos\alpha},\quad 0<\alpha<\frac{\pi}{2},$$

was proved in Problem 115. Prove the second half in a way that could have been understood before the invention of calculus.

168. *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, N.Y.*

If a, b, c are the sides of a triangle ABC, t_a, t_b, t_c are the angle bisectors, and T_a, T_b, T_c are the angle bisectors extended until they are chords of the circle circumscribing the triangle ABC, prove that

$$abc = \sqrt{T_aT_bT_ct_at_bt_c}\ .$$

169. *Proposed by Kenneth S. Williams, Carleton University.*

Prove that

$$\sqrt{5}+\sqrt{22+2\sqrt{5}} = \sqrt{11+2\sqrt{29}}+\sqrt{16-2\sqrt{29}+2\sqrt{55-10\sqrt{29}}}\,.$$

(This identity is due to Daniel Shanks, Naval Ship Research and Development Center, Bethseda, Maryland.)

170. *Proposed by Leroy F. Meyers, The Ohio State University.*

Is it possible to partition the plane into three sets A, B, and C (so that each point of the plane belongs to exactly one of the sets) in such a way that

(a) a counterclockwise rotation of 120° about some point P takes A into B, and

(b) a counterclockwise rotation of 120° about some point Q takes B into C?

* * *

LETTER TO THE EDITOR

Dear editor:

Enclosed are several solutions and comments to recent problems in EUREKA. I trust that you reply when asked about the origin of this publication, "EUREKA!? I have founded it!"

CLAYTON W. DODGE,
University of Maine at Orono.

S O L U T I O N S

115. [1976: 25, 98, 111] *Proposed by Viktors Linis, University of Ottawa.*

Prove the following inequality of Huygens:

$$2\sin\alpha + \tan\alpha \geq 3\alpha, \qquad 0 \leq \alpha < \frac{\pi}{2}.$$

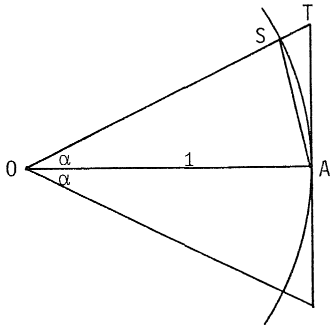
IV. *Solution by Leroy F. Meyers, The Ohio State University.*

The inequalities

$$\sin\alpha > \alpha - \frac{\alpha^3}{6} \text{ and } \tan\alpha > \alpha + \frac{\alpha^3}{3} \text{ if } 0 < \alpha < \frac{\pi}{2} \tag{1}$$

used in solution I [1976: 98] can be proved trigonometrically, hence (with rearrangement) geometrically. First we obtain

$$\sin\alpha < \alpha < \tan\alpha \text{ if } 0 < \alpha < \frac{\pi}{2}$$



by comparing the areas of triangle OAS, sector OAS, and triangle OAT (see figure). Now trigonometry gives

$$\sin 3\beta = 3\sin\beta - 4\sin^3\beta$$

and

$$\tan 3\beta = 3\tan\beta + \frac{8\tan^3\beta}{1 - 3\tan^2\beta}$$

(the latter holding only if 3β is not a multiple of $\frac{\pi}{2}$). Hence, if $0 < \beta < \frac{\pi}{6}$, then

$$3\sin\beta - \sin 3\beta = 4\sin^3\beta < 4\beta^3$$

and

$$\tan 3\beta - 3\tan\beta = \frac{8\tan^3\beta}{1 - 3\tan^2\beta} > 8\beta^3,$$

from which we obtain

$$(3\beta - \sin 3\beta) - 3(\beta - \sin\beta) < 4\beta^3$$

and

$$(\tan 3\beta - 3\beta) - 3(\tan\beta - \beta) > 8\beta^3.$$

Hence, if $0 < \alpha < \frac{\pi}{2}$ and n is a positive integer, then

$$3^{n-1}\left(\frac{\alpha}{3^{n-1}} - \sin\frac{\alpha}{3^{n-1}}\right) - 3^n\left(\frac{\alpha}{3^n} - \sin\frac{\alpha}{3^n}\right) < 3^{n-1}\cdot 4\left(\frac{\alpha}{3^n}\right)^3 = \frac{4\alpha^3}{3^{2n+1}}$$

and

$$3^{n-1}\left(\tan\frac{\alpha}{3^{n-1}} - \frac{\alpha}{3^{n-1}}\right) - 3^n\left(\tan\frac{\alpha}{3^n} - \frac{\alpha}{3^n}\right) > 3^{n-1}\cdot 8\left(\frac{\alpha}{3^n}\right)^3 = \frac{8\alpha^3}{3^{2n+1}}.$$

Now $3^n\left(\frac{\alpha}{3^n} - \sin\frac{\alpha}{3^n}\right)$ is twice the area of the region between an arc of length α and an inscribed regular polygonal arc with 3^n sides, and $3^n\left(\tan\frac{\alpha}{3^n} - \frac{\alpha}{3^n}\right)$ is the area of the region between an arc of length 2α and a circumscribed regular polygonal arc with 3^n sides. Both of these areas must approach 0 as n increases without bound. Hence summation of the last two inequalities as n goes from 1 to $+\infty$ yields

$$\alpha - \sin\alpha < \sum_{n=1}^{+\infty} \frac{4\alpha^3}{3^{2n+1}} = \frac{\alpha^3}{6}$$

and

$$\tan\alpha - \alpha > \sum_{n=1}^{+\infty} \frac{8\alpha^3}{3^{2n+1}} = \frac{\alpha^3}{3},$$

from which the desired inequalities follow immediately.

Except for notation and terminology, the above proof uses methods which were quite likely to have been known to Huygens in 1654. In addition to trigonometry, there are two limiting processes used: (1) the method of exhaustion (like Euclid XII • 2) for showing that the areas approach 0 as n increases without bound; and (2) the summation of an infinite geometric series with ratio $\frac{1}{9}$, which certainly was known to Archimedes, since a similar summation was used in his quadrature of a parabolic segment.

Editor's comment.

Proofs of the two inequalities in (1) can also be found in Hobson [1]. The proof of the first as given by Hobson is substantially the same as the one given here, but the proof of the second is quite different. The above solver himself belatedly discovered this reference and wrote to tell me about it. Thus we have another example of the frequently occurring phenomenon of rediscovery in mathematics.

I was fortunate enough to meet our solver, Leroy F. Meyers, in Toronto a few weeks ago at the summer meeting of the AMS-MAA. He had brought with him to show me Volume 12 of the *Oeuvres Complètes* of Huygens, a 22-volume set printed in the period 1888-1950 for *La Société Hollandaise des Sciences* by Martinus Nijhoff, La Haye. The set contains the complete Latin text of Huygens accompanied by a French translation. It is printed on beautifully watermarked paper, with the watermark *Christiaan Huygens* in italic type more than one inch high. Truly a bibliophile's delight.

With solution IV above and the proposer's comment following solution III [1976: 112], we have "pre-calculus" proofs of the first half of the double Snellius-Huygens inequality mentioned in the editor's comment in [1976: 99]. Some readers may be interested in trying to find a "pre-calculus" proof of the second half, and I invite you to do so in Problem 167 on page 136 of this issue.

REFERENCE

1. E.W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Dover, 1957, pp. 126-127.

120. [1976: 26, 103] *Proposed by John A. Tierney, United States Naval Academy.*

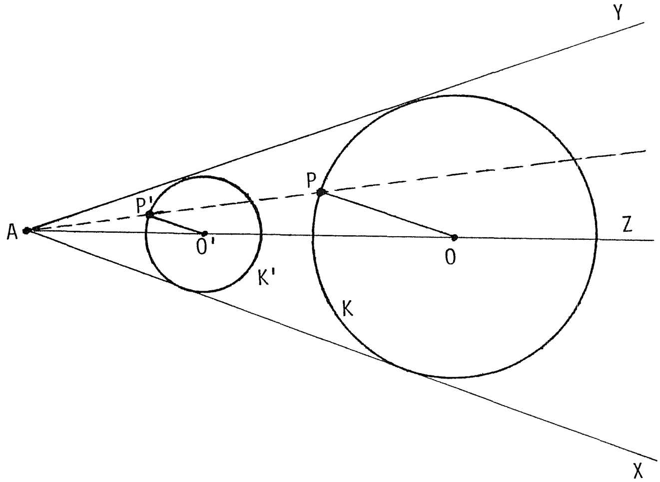
Given a point P inside an arbitrary angle, give a Euclidean construction of the line through P that determines with the sides of the angle a triangle

(a) of minimum area;

(b) of minimum perimeter.

II. *Comment by Dan Sokolowsky, Yellow Springs, Ohio.*

In part (b) of this problem, it becomes necessary to construct a circle tangent to the two sides of an $\angle XAY$ and passing through a given point P inside the angle. The construction given by the solver in solution I(b) is unnecessarily complicated. Here is a simpler one:



Take any point O' on the bisector AZ of $\angle XAY$ as center and construct a circle K' tangent to the sides of $\angle XAY$ (see figure). Draw ray AP and let P' denote the intersection of AP and K' which is nearest A. Construct a line through P parallel to O'P' meeting AZ at O. Then O is the centre of the required circle K through P tangent to the sides of $\angle XAY$. The proof follows easily from similar figures.

125. [1976: 41, 120] *Proposé par Bernard Vanbrugghe, Université de Moncton.*

A l'aide d'un compas seulement, déterminer le centre inconnu d'un cercle donné.

Editor's comment.

Duff Butterill, Ottawa Board of Education, sent in a late solution with a construction for which he gave the reference: Daus, *College Geometry*, Prentice-Hall, 1941. He also gave a geometric proof. Kenneth S. Williams, Carleton University, sent in an analytic proof of the same construction. The construction requires the drawing of six circles with Euclidean (collapsible) compasses. It is essentially the same as the construction given on pp. 40-42 of A.N. Kostovskii, *Geometrical Constructions Using Compasses Only*, Blaisdell, 1961, a reference I gave in my earlier comment [1976: 122].

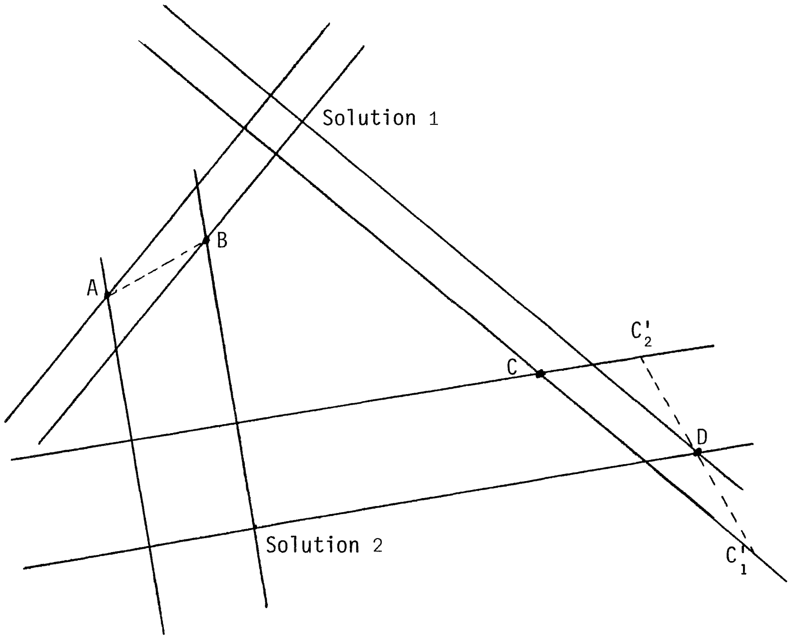
A brief but interesting discussion of geometry of the compasses, and of this problem in particular, can be found on pp. 122-124 of Dan Pedoe's *A Course of Geometry for Colleges and Universities*, Cambridge University Press, 1970.

127. [1976: 41, 124] *Proposed by Viktors Linis, University of Ottawa.*

A, B, C, D are four distinct points on a line. Construct a square by drawing two pairs of parallel lines through the four points.

III. *Comment by Clayton W. Dodge, University of Maine at Orono.*

Solution I [1976: 124] is indeed the general solution to the problem of finding a square whose four sides pass through four given coplanar points. That is, if two parallel sides pass through A and B, and the other pair of parallel sides pass through C and D, then locate point C' so that C'D is equal in length and perpendicular to AB (see figure). There are two such points C'. Then CC' is one side of the square, so draw a parallel to CC' through D and draw perpendiculars through A and B.



Proof. A 90° rotation carries A, B, and the two sides through A and B into D, C', and the two sides through C and D.

128. [1976: 41, 125] *Late solution:* KENNETH S. WILLIAMS, *Carleton University.*

131. [1976: 67] *Proposé par André Bourbeau, École Secondaire Garneau.*

Soit p un nombre premier ≥ 7. Si $p^{-1} = 0.\dot{a}_1 a_2 \ldots \dot{a}_k$, montrer que l'entier $N = \overline{a_1 a_2 \ldots a_k}$ est divisible par 9.

Solution by Kenneth S. Williams, Carleton University.

Let p be a prime ≥ 7. As $p \neq 2, 5$, p^{-1} does indeed have a periodic decimal expansion of the form

$$p^{-1} = 0.\dot{a}_1 a_2 \ldots \dot{a}_k.$$

Hence with $N = a_1 a_2 \ldots a_k$ (in decimal notation) we have

$$p^{-1} = N\left(\frac{1}{10^k} + \frac{1}{10^{2k}} + \ldots\right) = \frac{N}{10^k - 1},$$

that is, $10^k - 1 = Np$. Now $3^2 = 9 = 10 - 1 \mid 10^k - 1$, so $3^2 \mid Np$. But $p \neq 3$; hence $3^2 \mid N$, that is, $9 \mid N$.

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* G.D. KAYE, *Department of National Defence;* F.G.B. MASKELL, *Algonquin College; and the proposer.*

Editor's comments.

1. If the cycle length k of p^{-1} is even, that is, if

$$N = a_1 a_2 \ldots a_r a_{r+1} \ldots a_{2r},$$

(this occurs, in particular, if 10 is a primitive root of p, when $k = p - 1$), then we have the stronger result

$$a_1 + a_{r+1} = a_2 + a_{r+2} = \ldots = a_r + a_{2r} = 9.$$

For a discussion and proof of this statement see, for example, *Higher Algebra*, by S. Barnard and J.M. Child, Macmillan, London, 1969, pp. 439-443.

2. Essentially the same problem appeared as Problem 366 in the Spring 1976 issue of the *Pi Mu Epsilon Journal*, proposed by Richard Field, Santa Monica, California, but a solution has not yet been published in the *Journal*.

This seems a good place to mention that the *Pi Mu Epsilon Journal*, the official publication of the honorary mathematical fraternity, published twice a year (Spring and Fall) at the University of Oklahoma, is a remarkably interesting publication at about the same mathematical level as EUREKA. It has a vigorous problem section (about 25 pages per issue). The problem editor is Leon Bankoff, Los Angeles, California, an old friend, contributor, and booster of EUREKA. Subscriptions to the *Journal* are open to all. For more information write to David C. Kay, Editor, *Pi Mu Epsilon Journal*, 601 Elm, Room 423, The University of Oklahoma, Norman, Oklahoma 73069.

132. [1976: 67] *Proposed by Léo Sauvé, Algonquin College.*

If $\cos\theta \neq 0$ and $\sin\theta \neq 0$ for $\theta = \alpha, \beta, \gamma$, prove that the normals to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the points of eccentric angles α, β, γ are concurrent if and only if

$$\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0.$$

Solution by F.G.B. Maskell, Algonquin College.

From the parametric equations of the ellipse

$$x = a\cos\theta, \quad y = b\sin\theta, \quad -\pi < \theta \leq \pi,$$

the equation of the normal at θ is

$$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta, \tag{1}$$

or, in terms of $t = \tan\frac{\theta}{2}$, $-\pi < \theta < \pi$,

$$2atx - b(1 - t^2)y = \frac{2(a^2 - b^2)t(1 - t^2)}{1 + t^2}, \quad t \in R$$

and $y = 0$ for $\theta = \pi$.

If we set $t_1 = \tan\frac{\alpha}{2}$, $t_2 = \tan\frac{\beta}{2}$, $t_3 = \tan\frac{\gamma}{2}$, then the normals at α, β, γ are concurrent if and only if

$$\begin{vmatrix} t_1(1 + t_1^2) & 1 - t_1^4 & t_1(1 - t_1^2) \\ t_2(1 + t_2^2) & 1 - t_2^4 & t_2(1 - t_2^2) \\ t_3(1 + t_3^2) & 1 - t_3^4 & t_3(1 - t_3^2) \end{vmatrix} = 0$$

or

$$\Delta = \begin{vmatrix} t_1 + t_1^3 & 1 - t_1^4 & t_1 \\ t_2 + t_2^3 & 1 - t_2^4 & t_2 \\ t_3 + t_3^3 & 1 - t_3^4 & t_3 \end{vmatrix} = 0.$$

It is evident that $t_2 - t_3$, $t_3 - t_1$, $t_1 - t_2$ are factors of Δ. Subtracting rows to obtain the common factors and expanding the resulting determinant, we obtain

$$(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)(\Sigma t_1 - t_1 t_2 t_3 \Sigma t_2 t_3) = 0.$$

The fourth factor equated to 0 is equivalent to

$$\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0 \tag{2}$$

since, in terms of t_1, t_2, t_3, (2) is equivalent to

$$\Sigma(1 + t_1^2)\{t_2(1 - t_3^2) + t_3(1 - t_2^2)\} = 0,$$

which simplifies to

$$\Sigma t_1 - t_1 t_2 t_3 \Sigma t_2 t_3 = 0.$$

Since $-\pi < \alpha, \beta, \gamma < \pi$, the equations $t_2 - t_3 = 0$, $t_3 - t_1 = 0$, $t_1 - t_2 = 0$ are equivalent to $\beta = \gamma$, $\gamma = \alpha$, $\alpha = \beta$. Thus, *provided that no two of α, β, γ are equal*, the necessary and sufficient condition for concurrency is given by (2). It is easy to verify that if exactly one of α, β, γ is equal to π (excluded from the above analysis), then (2) is still a necessary and sufficient condition for concurrency. The conditions $\sin\theta \neq 0$ and $\cos\theta \neq 0$ for $\theta = \alpha, \beta, \gamma$, given by the proposer, are not needed.

Also solved by the proposer.

Editor's comment.

It is clear that we have concurrency of normals but (2) does not always hold if α, β, γ are all equal. But I believe (which means I can only prove it by waving my arms) that if exactly two of α, β, γ are equal, say $\alpha = \beta \neq \gamma$, then (2) remains a necessary and sufficient condition for concurrency, and that the intersection of the normals is then the centre of curvature at α. I hope some reader can formulate a closely reasoned argument (perhaps based on continuity) to either prove or disprove this conjecture. If it is disproved, then solution II of Problem 103 [1976: 75] becomes invalid.

I have located this problem, without solution, in [2] and [3]. In [2] the following valuable hint is given: use the product

$$P = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin 2\alpha \\ \sin\beta & \cos\beta & \sin 2\beta \\ \sin\gamma & \cos\gamma & \sin 2\gamma \end{vmatrix} \begin{vmatrix} \Sigma\cos\alpha - \cos\Sigma\alpha & 0 & 1 \\ \Sigma\sin\alpha + \sin\Sigma\alpha & 1 & 0 \\ -1 & 0 & 0 \end{vmatrix}. \qquad (3)$$

The rest of the solution can easily be imagined. It is clear from (1) that the vanishing of the first determinant in (3) (and hence of the product P since the second determinant has value 1) is a necessary and sufficient condition for concurrency of normals. One can then show that

$$P = 4\{\Sigma\sin(\beta+\gamma)\}\,\Pi\sin\tfrac{1}{2}(\beta-\gamma), \qquad (4)$$

and (2) follows when α, β, γ are all distinct.

I have also found in [1] an interminable solution showing in painful detail how the first determinant in (3) by itself can be transformed into (4).

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133. [1976: 67] *Proposed by Kenneth S. Williams, Carleton University.*

Let f be the operation which takes a positive integer n to $\frac{1}{2}n$ (if n even) and to $3n+1$ (if n odd). Prove or disprove that any positive integer can be reduced to 1 by successively applying f to it.

Example: $13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$.

(This problem was shown to me by one of my students.)

I. *Comment by Charles W. Trigg, San Diego, California.*

It should be noted that the sequences developed by this routine actually terminate in the three-member loop 4, 2, 1.

This alluring algorithm was perceived by Lothar O. Collatz (now a professor at the University of Hamburg) during his student days prior to World War II [8, 9]. Since that time it has been called periodically to public attention—by Collatz himself in 1950 at Harvard University [9], and by H.S.M. Coxeter in a 1970 lecture [6]. In this lecture Coxeter offered a prize of fifty dollars for the first valid proof, or a prize of a hundred dollars for the first counter-example.

In 1972, Ogilvy [12] listed the conjecture among "unsolved problems for the amateur." In the same year, Gardner [7] included it in a discussion of "looping problems." In 1975, Rogers [14] listed it as one of the more difficult of his problems for pocket computers. Other treatments have appeared in professional publications in the course of which the discovery of the routine has been attributed to a variety of persons.

Computers have been used to attack the problem. Coxeter [6] reported that all positive integers, N, had been tested up to 500,000 without finding an exception. Michael Beeler and associates [5] have established convergence for all $N < 6 \times 10^7$, others [1] have carried on up to 2×10^8, and John Selfridge [8] to 10^9. It has been stated also [2] that "A.S. Frankel at the Weizmann Institute has used a combination of mathematical and computing techniques to show that the process converges to 1 for all values of $N < 10^{40}$."

It is not necessary to carry the sequence for any N beyond a term $< N$. Thus it is not necessary to test any even values, nor odd values of the form $4k+1$, $16k+3$, or $128k+7$. This is evident from the sequences:

$4k+1,\ 12k+4,\ 6k+2,\ 3k+1$;

$16k+3,\ 48k+10,\ 24k+5,\ \ldots,\ 18k+4,\ 9k+2$; and

$128k+7,\ 384k+22,\ 192k+11,\ \ldots,\ 162k+10,\ 81k+5$.

These have led Curtis Gerald [2] to speculate that the fraction of all numbers thus shown to be convergent without testing is

$$1/2 + 1/4 + 1/16 + 1/128 + \ldots + 1/B_{n-1} + 1/B_n, \text{ where } B_{n+1}/B_n = 2^n.$$

Designate the number of operations necessary to convert N into 1 by I_N, the index of N. Thus the sequence belonging to N will contain $I_N + 1$ terms including N itself. Let N_m be the largest integer in the sequence.

In some cases, the sequence builds up to a relatively large N_m before crunching down to 1. Richard Andree [2] has observed that $I_{27} = 111$ and $N_m/N = 9232/27 \doteq 342$, whereas another $N_m/N = 1024984918960/200000342 \doteq 5125$.

Certain I values occur more frequently than others. For example, in the range of N from 90,000 through 94,999 over half of the I values are 84, 115, 133, 146, and 177 [1].

Once a power of 2 occurs in a sequence, the convergence is immediate. In a majority of the cases the first power of 2 encountered is 16 [11].

There exist many strings of successive values of N with equal I_N's. In the following table, the smallest N in the first appearance of a string of each of the stated lengths is given [15].

String length	Smallest N	I_N	String length	Smallest N	I_N
2	12	9	9	1680	42
3	28	18	10	4722	59
4	314	37	11	6576	137
5	98	25	13	3982	51
6	840	41	14	2987	48
7	943	36	17	7083	57
8	1494	47			

In the following table the final N of a string of each stated length is given. These are *not* the first appearances of such strings in all cases [3]:

String length	Final N	String length	Final N
12	131917	24	221208
15	134270	25	57370*
16	243839	26	393242
18	137169	27	252574
19	447262	29	331806
20	454461	30	524318
21	152216	32	913350
22	212181	35	1032909
23	362520	40	596349

*Wardrop [16] states that this is the first appearance for a string length of 25.

Where consecutive N's have the same I their sequences merge at a node to form a tree, as in

12	6	3	10						
13	40	20	10	5	16	8	4	2	1

Mary Krimmel [10] has suggested that all pairs of the form $8n + 4$ and $8n + 5$ are twins with equal I's. This is evident from the sequences

$$\begin{array}{cccc} 8n+4 & 4n+2 & 2n+1 & 6n+4 \\ 8n+5 & 24n+16 & 12n+8 & 6n+4. \end{array}$$

That is, the two sequences merge at the third operation.

Indeed, the sequences for

$16n + 2$ and $16n + 3$ merge at the 5th operation,
$32n + 22$ and $32n + 23$ merge at the 7th operation,
$64n + 14$ and $64n + 15$ merge at the 9th operation,
$128n + 94$ and $128n + 95$ merge at the 11th operation,
$256n + 62$ and $256n + 63$ merge at the 13th operation,
$512n + 382$ and $512n + 383$ merge at the 15th operation,
$1024n + 254$ and $1024n + 255$ merge at the 17th operation [15].

There are many more of these generalized two-strings. Members of the various two-strings may upon occasion coincide with members of triads, quartets, or longer strings having equal I's.

The sequences for $32n + 36$ and $32n + 37$ merge at the 3rd operation, and continue on to the 6th operation to merge with the sequence for $32n + 38$. The smallest member of other three-strings are $64n + 44$ (merging at operation 8), $128n + 28$ (merging at operation 10), $256n + 128$ (merging at operation 12), etc.

$2048n + 314$ is the smallest member of a string quartet that have completely merged at the 15th operation.

$256n + 98$ is the smallest member of a string quintet that have completely merged at the 10th operation [15].

The beguilingly simple calculations involved in this routine have enticed many amateurs to give "proofs" of convergence for all positive N. Most of these "proofs" could be applied with equal validity to the modified routine where 1 is subtracted rather than added ($\div 2$, $3x - 1$). The catch is that in this case there are three different loops in which the sequences terminate:

2, 1, 2, 1, ... ;
5, 14, 7, 20, 10, 5,...; and
17, 50, 25, 74, 37, 110, 55, 164, 82, 41, 122, 61, 182, 91, 272,
136, 68, 34, 17, ... [5].

The difficulty of proof has engaged the attention of number theorists and other qualified mathematicians, but no one has yet devised a satisfactory proof or strategy for proof. Collatz's algorithm is a worthy companion of Goldbach's conjecture.

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II. *Comment by Clayton W. Dodge, University of Maine at Orono.*

This problem has surfaced at various meetings for the past few years and, to the best of my knowledge, has firmly resisted solution. With the aid of a Texas Instruments SR-52 calculator I have verified the theorem for all integers n from 1 to 1200 and I have also calculated the number of steps $S(n)$ to reach 1. Thus $S(1)=0$, $S(2)=1$, and $S(13)=9$. Some rather curious patterns appear in the tabulated results. Although clearly $S(2n) = S(n)+1$, there would seem to be no relation between $S(n)$ and $S(n+1)$. Surprisingly, relatively few values of $S(n)$ appear in any block of values of n. Table 1 lists these values for the reasonably typical interval from $n=901$ to 1000. Table 2 lists the only 16 values of $S(n)$ and their frequencies as found in that interval. Observe especially that $S(n)=36$ for $943 \le n \le 949$, seven consecutive values of n all having the same value of $S(n)$. Clearly, there is much more to this problem than appears on the surface.

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[1]Gardner [7] calls this HAKMEM (short for "Hacker's Memo"). Editor.

	901 -910	911 -920	921 -930	931 -940	941 -950	951 -960	961 -970	971 -980	981 -990	991 -1000
1	54	41	129	36	129	28	49	36	23	98
2	54	36	36	85	129	36	49	98	142	111
3	116	129	67	85	36	28	49	98	142	93
4	15	129	129	85	36	36	23	142	49	23
5	67	129	129	129	36	28	23	142	23	23
6	15	36	129	23	36	54	23	23	49	49
7	54	36	116	173	36	54	142	98	36	49
8	15	129	23	23	36	54	98	49	49	49
9	15	129	129	85	36	129	49	49	49	49
10	41	36	36	129	28	23	98	23	98	111

Table 1. Values of $S(n)$ for $901 \leq n \leq 1000$.

$S(n)$	freq.	$S(n)$	freq.	$S(n)$	freq.	$S(n)$	freq.
15	4	41	2	85	4	116	2
23	13	49	14	93	1	129	15
28	4	54	6	98	7	142	5
36	18	57	2	111	2	173	1

Table 2. Values of $S(n)$ and their frequencies of occurrence in the interval $901 \leq n \leq 1000$.

III. *Comment by Leroy F. Meyers, The Ohio State University.*

A problem similar to this one was mentioned by Stephen D. Isard (Department of Machine Intelligence and Perception, University of Edinburgh) and Arnold M. Zwicky (Linguistics, The Ohio State University) in their paper in some publication of the Special Interest Group on Automata and Computability Theory (SIGACT) of the Association for Computing Machinery (ACM) in 1970 or earlier. (I have a copy of the paper, mimeographed or offset printed, but no reference that is legible. I can just barely read "SIGACT" and the page numbers 11-19. I received it in 1970.)

Let $f(n) = n/3$ if n is divisible by 3, and let $g(n) = 2n+1$. Can all numbers not congruent to 2 modulo 3 be reduced to 1 by a succession of operations f and g? (The problem is stated in another way on p. 14. The title of the paper is: Three open questions in the theory of one-symbol Smullyan systems.) Some fairly small numbers require long sequences of operations. For example:

$$7 \to 15 \to 31 \to 63 \to 21 \to 43 \to 87 \to 175 \to 351 \to 117 \to 39 \to 13 \to 27 \to 9 \to 3 \to 1.$$

The portion $63 \to \ldots \to 39$ may be done alternately as

$$63 \to 127 \to 255 \to 85 \to 171 \to 57 \to 19 \to 39.$$

Editor's comments.

1. We all know now what I should have known earlier, that this is a famous unsolved problem. But then every problem was once unsolved. The recent announcement, with great fanfare at the 1976 summer meeting of mathematical societies in Toronto, that the four-colour problem had finally been solved about one hundred twenty-five years after it had been proposed, leads us to hope that this problem too may eventually be solved, although perhaps not in the next issue of EUREKA! I give below some information I've been able to dredge up that is not included in Trigg's encyclopedic comment I, in the hope that some readers will take it from there. Perhaps some time soon one of them will be able to shout: *Eureka!*

2. In Trigg's reference [12], Ogilvy states that generalization, at least in some directions, is impossible. He notes that if the Collatz algorithm is applied to negative integers the result, in absolute values, is the same as using positive integers with the revised rule $3n-1$ (for odd integers) instead of $3n+1$. He adds that all negative integers to -10^8 were found to enter one of the three loops mentioned by Trigg at the end of his comment I. Ogilvy also states that if the rule $3n+1$ is changed to $5n+1$ for odd positive integers, then there exist cycles other than 4, 2, 1. Example:

$$13,\ 66,\ 33,\ 166,\ 83,\ 416,\ 208,\ 104,\ 52,\ 26,\ 13,\ \ldots$$

In Trigg's reference [16], Wardrop baldly states that *every positive integer* can be changed to one by applying Collatz's algorithm. (Hey! ... Psst! ... Wardrop! Do you know something we don't know?)

3. An unsigned problem and (partial) solution in [17] contain the following information about this problem.

Define the function $f : N \to N$ by

$$f(n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even} \\ \dfrac{3n+1}{2}, & \text{if } n \text{ is odd.} \end{cases} \qquad (1)$$

Since $3n+1$ is always even when n is odd, the second part of the definition merely ensures that n will meet the same ultimate fate but in fewer steps than in the original algorithm.

Define the iterates of f recursively by

$$f^0 = I, \quad f^{i+1} = f^i \circ f, \quad i = 0,1,2,\ldots,$$

where I is the identity function on N, and let

$$M = \{n \in N \mid \exists\, k \in N,\ f^k(n) = 1\}, \quad M' = N - M.$$

The conjecture we seek to prove is then that M' is empty.

For $n \in M$, let $\theta(n)$ be the smallest positive integer such that $f^{\theta(n)}(n) = 1$.

The following facts are then obvious or easily established:

(i) If $2n \in M$, then $n \in M$ and $\theta(2n) = \theta(n)+1$.

(ii) For every positive integer i, $\theta(2^i) = i$.

(iii) If $2n+1 \in M$, then $3n+2 \in M$ and $\theta(2n+1) = \theta(3n+2)+1$.

(iv) If $4n+1 \in M$, then $3n+1 \in M$ and $\theta(4n+1) = \theta(3n+1)+2$.

Suppose M' is not empty, and let α be its smallest element. It is clear from the choice of α that we must have $f^k(\alpha) \geq \alpha$ for all k. It follows from information given in Trigg's comment I that α, if it exists, is greater than 10^{40} and is not of any of the forms $2k$, $4k+1$, $16k+3$, or $128k+7$.

4. Since all attempts at solving this problem so far have failed, I hesitantly suggest the following approach which *may*, if it has not already been tried, bring us one step closer to a solution.

The function $h : R \to R$ defined by

$$h(x) = \tfrac{1}{4}\{4x+1-(2x+1)\cos \pi x\} \tag{2}$$

is an extension of the function f defined in (1), since $h(x) = f(x)$ whenever $x \in N$.

If one could express $h^k(x)$, the kth iterate of $h(x)$, explicitly in terms of k and x, there is a possibility that the form of $h^k(x)$ might enable us to determine if, for a given positive integer x, there exists a positive integer k such that $h^k(x) = 1$. It might even give us valuable information on the behaviour of the iterates of (2) for nonintegral values of x.

Let g be a function whose iterates g^k can be determined explicitly, and let ϕ (or ψ) be such that

$$\phi h = g\phi \quad (\text{or } h\psi = \psi g),$$

(where ϕh means $\phi \circ h$, etc.); then

$$h^2 = (\phi^{-1}g\phi)(\phi^{-1}g\phi) = \phi^{-1}g^2\phi \quad (\text{or } h^2 = \psi g^2\psi^{-1}),$$

and in general

$$h^k = \phi^{-1}g^k\phi \quad (\text{or } h^k = \psi g^k\psi^{-1}).$$

Since g^k is known, so is h^k. The functions h and g are then said to be conjugates under ϕ (or ψ). If $g(x) = x+1$, $\phi(x)$ is known as the Abel function, while if $g(x) = cx$ $(c \neq c^2)$, $\phi(x)$ is known as the Schroeder function for $h(x)$.

For example, if $h_1(x) = 2x^2-1$, $\psi(x) = \cos x$, $g(x) = 2x$, then $h_1\psi = \psi g$ and we find

$$h_1^k(x) = \cos(2^k \arccos x), \quad -1 \leq x \leq 1.$$

Possibly finding an appropriate g and ϕ (or ψ) will turn out to be as difficult as the original problem, but it might be worth a try.

More information about the process of explicit iteration can be found in [18].

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134. [1976: 68] *Proposed by Kenneth S. Williams, Carleton University.*

ABC is an isosceles triangle with $\angle ABC = \angle ACB = 80°$. P is the point on AB such that $\angle PCB = 70°$. Q is the point on AC such that $\angle QBC = 60°$. Find $\angle PQA$.

(This problem is taken from the 1976 Carleton University Mathematics Competition for high school students.)

I. *Solution by Clayton W. Dodge, University of Maine at Orono.*

By the law of sines (see Figure 1)

$$\frac{\sin 70°}{BP} = \frac{\sin 30°}{BC} \quad\text{and}\quad \frac{\sin 80°}{BQ} = \frac{\sin 40°}{BC},$$

so that

$$BP = \frac{BC \sin 70°}{\sin 30°} = 2BC \sin 70°$$

and

$$BQ = \frac{BC \sin 80°}{\sin 40°} = 2BC \cos 40° = 2BC \sin 50°.$$

Since

$$\sin 70° - \sin 50° = 2 \cos 60° \sin 10° = \sin 10°$$

and

$$\sin 70° + \sin 50° = 2 \sin 60° \cos 10° = \sqrt{3} \cos 10°,$$

it follows that

$$\frac{BP - BQ}{BP + BQ} = \frac{1}{\sqrt{3}} \tan 10°.$$

Now let $u = \angle BPQ$ and $v = \angle BQP$. Applying the law of tangents to ΔBPQ, we get

$$\tan\frac{v-u}{2} = \frac{BP - BQ}{BP + BQ} \cot 10° = \frac{1}{\sqrt{3}}.$$

Thus $\frac{v-u}{2} = 30°$, and since $\frac{v+u}{2} = 80°$, it follows that $u = 50°$ and $x = u - 20° = 30°$.

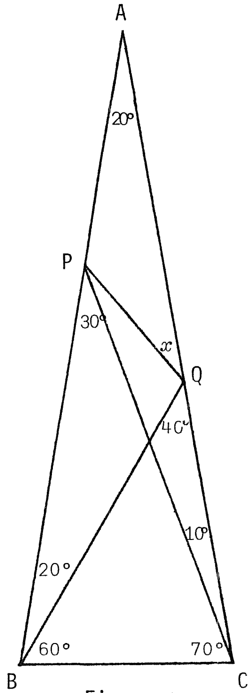

Figure 1

II. *Comment by Dan Eustice, The Ohio State University.*

I found a problem very similar to this one in Coxeter and Greitzer [1].

An interesting feature of this problem is that the given triangle belongs to a one-parameter family of triangles in each of which the angles 20°, 20°, 10° are as shown in Figure 2, assuming the point Q to be fixed (say at unit distance from A). For the angles α, β, γ, δ of the figure satisfy the system

$$\alpha + \beta = 160°,\ \alpha + \gamma = 150°,\ \beta + \delta = 140°,\ \gamma + \delta = 130°$$

which has solutions

$$(\alpha, \beta, \gamma, \delta) = (100° + c,\ 60° - c,\ 50° - c,\ 80° + c).$$

Given a choice of c, $-80° < c < 50°$, there is a uniquely determined ΔABC which satisfies the configuration of Figure 2, and furthermore this triangle need not be isosceles. In Figure 1, $c = 30°$ ($\beta = 30°$), and Figure 2 was constructed with $c = 0$ ($\beta = 60°$).

Also solved by DAN EUSTICE, *The Ohio State University (solution as well);* G.D. KAYE, *Department of National Defence; and the proposer.*

Editor's comment.

The similar problem from Coxeter and Greitzer [1] mentioned by Eustice differs from our own only in that the angles PCB and QBC are 50° and 60° instead of 70° and 60°. But our own 70°-60° version of the problem appears considerably more difficult than the 50°-60° version, at least insofar as finding a purely geometric solution is concerned. (I would be curious to know how many high school students succeeded in solving this problem in the 1976 Carleton University Mathematics Competition.)

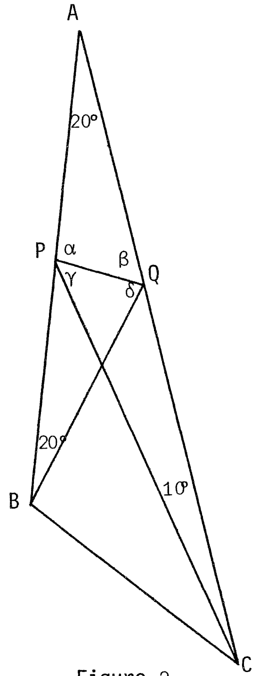

Figure 2

Coxeter and Greitzer give a fairly simple geometric proof of the 50°-60° problem in [1], and Honsberger [2], who calls it "an old chestnut" and a "hardy perennial," gives another ingenious geometric solution which he attributes to S.T. Thompson of Tacoma, Washington. But all solutions submitted for our own 70°-60° problem were trigonometric. It would be interesting to have an elegant purely geometric proof. I hope I am wrong

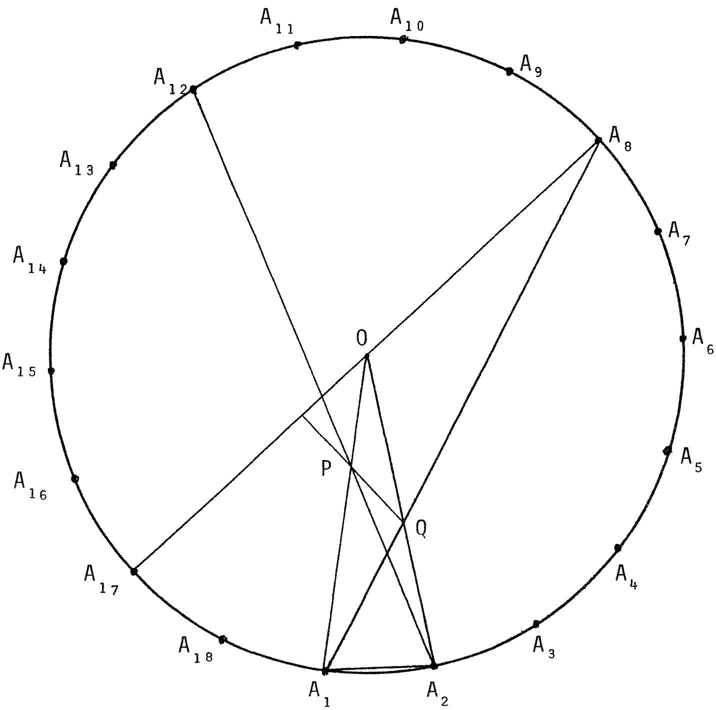

Figure 3

in suspecting that such a geometric proof will not be easy to find. I describe below one approach, based on the method used by Thompson in [2], which *may* lead to a geometric solution.

Divide a circle with centre O into 18 equal parts by the points A_1, A_2, ..., A_{18} (see Figure 3). Draw in the sides of ΔOA_1A_2. Draw chords A_2A_{12} and A_1A_8, meeting OA_1 and OA_2 in P and Q, respectively. It is easy to see that segment PQ subtends angles of 20° at A_1 and 10° at A_2, and so ΔOA_1A_2 is the one we are concerned with in this problem. Draw diameter A_8A_{17}. A geometric proof that $A_8A_{17} \perp PQ$ would lead to the immediate conclusion that $\angle PQO = 30°$.

I hope some readers will be able to complete this proof, or else send in an entirely different geometric proof.

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135. [1976: 68] *Proposed by Steven R. Conrad, B.N. Cardozo High School, Bayside, N.Y.*

How many 3×5 rectangular pieces of cardboard can be cut from a 17×22 rectangular piece of cardboard so that the amount of waste is a minimum?

I. *Solutions submitted independently by Clayton W. Dodge, University of Maine at Orono; G.D. Kaye, Department of National Defence; Charles W. Trigg, San Diego, California; and the proposer.*

The maximum number is 24 pieces. See Figure 1.

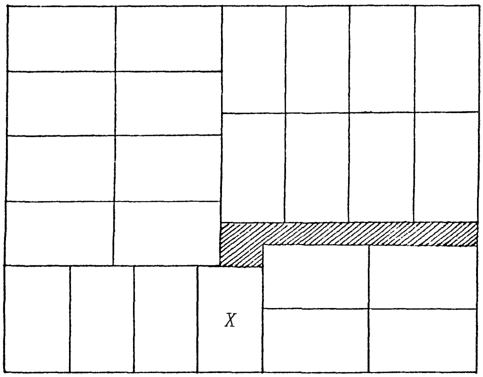


Figure 1

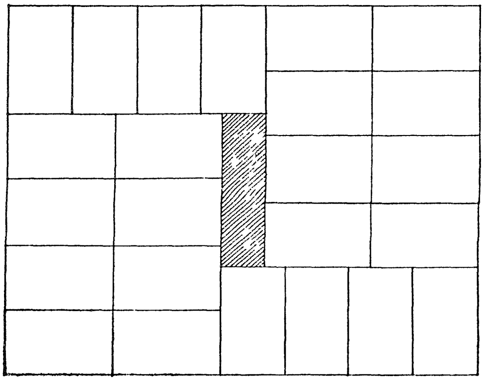


Figure 2

II. *Solutions submitted independently by F.G.B. Maskell, Algonquin College; and Kenneth S. Williams, Carleton University.*

The maximum number is 24 pieces. See Figure 2.

III. *Comment by Clayton W. Dodge, University of Maine at Orono.*

If the cards are to be cut by a stamping process, solutions are easily found (see Figures 1 and 2). If, however, they are to be cut by the usual paper knife, then one sees that there is no straight line on which to make the first cut. The layouts in Figures 1 and 2, then, are not satisfactory for such cutting.

It appears, but I have no proof, that one cannot find such a layout of 24 cards that can all be cut with a paper knife. That it can be done with 23 cards is seen by letting card X (in Figure 1) be waste and taking the first cut along the vertical line that goes through that card. Layouts of 23 cards that are more efficient for the cutter are easily found.

To minimize waste one might try to cut cards of different sizes from the sheet. By considering areas, one finds that the waste cannot be reduced to zero by cutting only two of the standard sizes 3×5, 4×6, and 5×8. However, Figure 3 allows zero waste using the maximum number of 3×5 cards, namely ten, along with six 4×6 cards and two 5×8 cards.

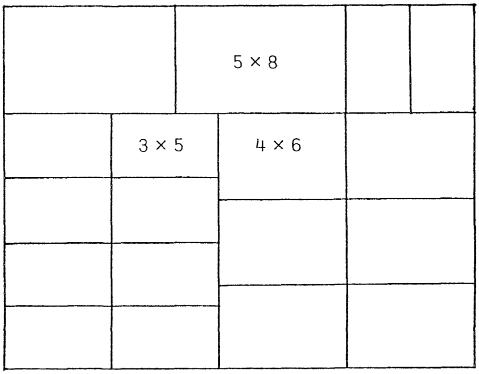


Figure 3

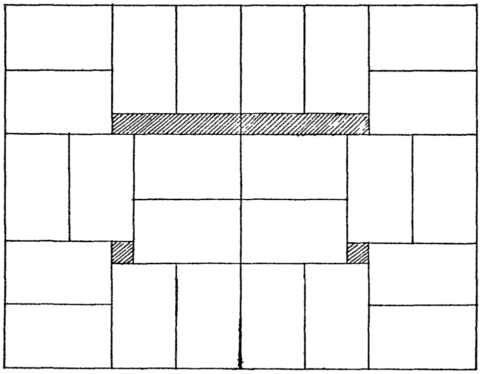


Figure 4

Editor's comments.

1. Every solver of course pointed out that 24 is the maximum number of 3×5 cards that can be cut, since the amount of waste is then 14 square units, less than the area of one 3×5 card.

2. We now have open for proof or disproof the interesting question raised by Dodge, whether there exists on a 17×22 card a layout of twenty-four 3×5 rec-

tangles all of which can be cut with a paper knife. While we wait for one of our ingenious readers to answer this question, let us consider the following easier related question: find on a 17×22 card a layout of twenty-four 3×5 rectangles of which as many as possible can be cut with a paper knife without spoiling any other. Figure 4 shows a layout in which eight of the twenty-four rectangles can be cut with a paper knife. Can any reader do better?

3. A natural question to ask in the present context is the following: when can an $a \times b$ rectangle be packed (i.e. no waste) with $c \times d$ rectangles? In [1] Honsberger attributes to David A. Klarner, and gives a proof of, the following two theorems which completely answer the question. The second theorem is an easy consequence of the first.

THEOREM 1. *An* $a \times b$ *rectangle can be packed with* $1 \times n$ *strips if and only if* n *divides* a *or* n *divides* b.

THEOREM 2. *An* $a \times b$ *rectangle can be packed with* $c \times d$ *rectangles if and only if either* (i) *each of* c, d *divides one of* a, b, *each a different one, or* (ii) *both* c *and* d *divide the same one of* a, b, *say* a, *and the other* (b) *is of the form* $b = cx + dy$, *for some nonnegative integers* x *and* y.

REFERENCE

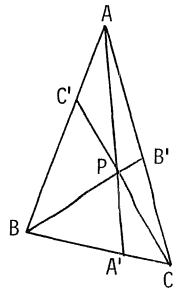
1. Ross Honsberger, *Mathematical Gems II*, The Mathematical Association of America, 1976, pp. 67-68, 172-175.

136. [1976: 68] *Proposed by Steven R. Conrad, B.N. Cardozo High School, Bayside, N.Y.*

In $\triangle ABC$, C' is on $\overline{AB}$ such that $AC':C'B = 1:2$ and B' is on $\overline{AC}$ such that $AB':B'C = 4:3$. Let P be the intersection of $\overline{BB'}$ and $\overline{CC'}$, and let A' be the intersection of $\overline{BC}$ and ray AP. Find $AP:PA'$.

Solution by Charles W. Trigg, San Diego, California.

By a corollary of Ceva's Theorem (see [1], for example), we have (see figure)



$$\frac{AP}{PA'} = \frac{AC'}{C'B} + \frac{AB'}{B'C} = \frac{1}{2} + \frac{4}{3} = \frac{11}{6}.$$

Also solved by RADFORD DE PEIZA, *Woburn C.I., Scarborough, Ont.;* CLAYTON W. DODGE, *University of Maine at Orono;* KENNETH S. WILLIAMS, *Carleton University; and the proposer.*

Editor's comment.

The theorem used in the above solution can also be found in [2], which is perhaps more easily accessible. Some solvers used a combination of Ceva's Theorem and Menelaus' Theorem to arrive at the answer. The proposer pointed out that the answer can also be obtained by using mass points (see [3]).

REFERENCES

1. Nathan Altshiller Court, *College Geometry*, Johnson Publishing Co., 1925, Section 245, p. 131.

2. Nathan Altshiller Court, *College Geometry*, Second Edition, Barnes and Noble, 1952, Section 342, p. 163.

3. Harry Sitomer and Steven R. Conrad, Mass Points, *Eureka*, Vol. 2 (1976), pp. 55-62.

137. [1976: 68] *Proposed by Viktors Linis, University of Ottawa.*

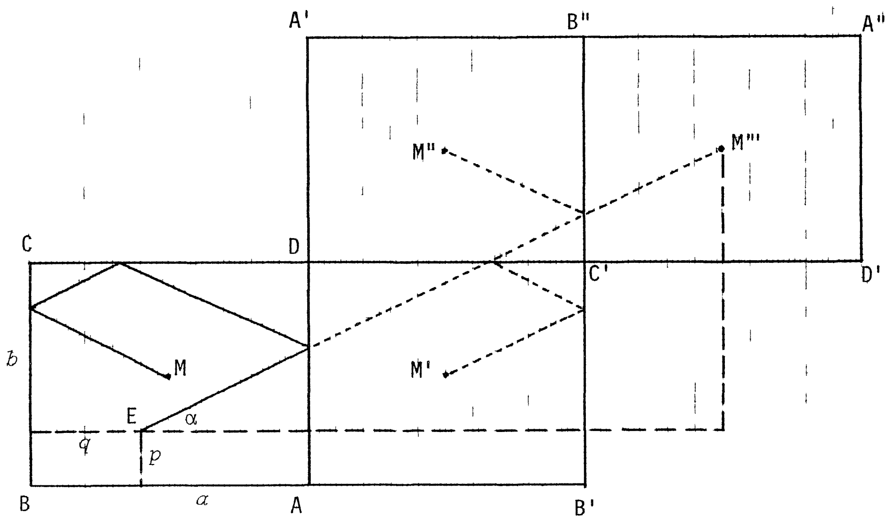
On a rectangular billiard table ABCD, where AB $= a$ and BC $= b$, one ball is at a distance p from AB and at a distance q from BC, and another ball is at the centre of the table. Under what angle α (from AB) must the first ball be hit so that after the rebounds from AD, DC, CB it will hit the other ball?

Solution by Charles W. Trigg, San Diego, California.

Sequentially reflect the table about AD, DC, and CB as shown in the figure. It is immediately obvious that $\tan\alpha = (3b/2 - p)/(5a/2 - q)$, so that

$$\alpha = \arctan\frac{3b - 2p}{5a - 2q},$$

provided the line EM‴ lies within the bounds of the figure and does not pass through M. Otherwise the shot is not possible.



Also solved by G.D. KAYE, *Department of National Defence; and* KENNETH S. WILLIAMS, *Carleton University.*

Editor's comment.

This problem is at first quite intriguing, but the ingenious reflection method used above shows that nothing but the most elementary Triggonometry is needed for its solution.

The simplicity of the solution is due more to the power of the reflection method than to the rectangular shape of the table, since the method is equally effective in solving the corresponding problem involving a table of arbitrary triangular shape (see Melzak [2]).

On the other hand, it is far from easy to solve a well-known similar problem involving a circular table. This problem, which is known as *Alhazen's Billiard Problem*, has engaged the attention of many famous mathematicians after Alhazen (ca. 965 - ca. 1039), in particular Huygens, Barrow, L'Hôpital, and Riccati. See Dörrie [1] for a solution and full discussion of Alhazen's Problem.

REFERENCES

1. Heinrich Dörrie, *100 Great Problems of Elementary Mathematics*, Dover, 1965, pp. 197-200.

2. Z.A. Melzak, *Companion to Concrete Mathematics*, Wiley, 1973, pp. 28-29.

138. [1976: 68] *Proposé par Jacques Marion, Université d'Ottawa.*

Soit $p(z) = z^n + a_1 z^{n-1} + \ldots + a_n$ un polynôme non constant tel que $|p(z)| < 1$ sur $|z| = 1$. Montrer que $p(z)$ a un zéro sur $|z| = 1$.

I. *Solution de Leroy F. Meyers, The Ohio State University.*

Posons $q(z) = z^n p(\frac{1}{z})$ si $z \neq 0$ et $q(0) = 1$. Puisque $\lim\limits_{z\to 0} q(z) = 1$, la fonction q est analytique sur le disque $|z| \leq 1$. D'après le théorème du maximum pour les fonctions analytiques, il existe un nombre z tel que $|z| = 1$ et que $|q(z)| \geq |q(0)| = 1$. Mais, pour un tel nombre, $|1/z| = 1$ et

$$|p(\frac{1}{z})| = |z^{-n} q(z)| = |q(z)| \geq 1 .$$

L'hypothèse "$|p(z)| < 1$ sur $|z| = 1$" étant fausse, on peut en déduire une conclusion quelconque.

II. *Solution by Kenneth S. Williams, Carleton University.*

Suppose there exists a nonconstant polynomial

$$p(z) = z^n + a_1 z^{n-1} + \ldots + a_n$$

such that $|p(z)| < 1$ on $|z| = 1$. Then

$$|z^n + (a_1 z^{n-1} + \ldots + a_n)| < |-z^n|$$

on $|z| = 1$, and so by Rouché's Theorem

$$z^n + (a_1 z^{n-1} + \ldots + a_n) - z^n \quad \text{and} \quad -z^n$$

have the same number of zeros (counted with respect to multiplicity), that is, $a_1 z^{n-1} + \ldots + a_n$ has n zeros, clearly a contradiction. Hence there do not exist polynomials $p(z)$ such as described in the hypothesis of this problem.

Also solved by KENNETH S. WILLIAMS, *Carleton University (second solution); and the proposer.*

Editor's comment.

The proposer pointed out that this problem can be found, exactly as given above (except, of course, in English), on p. 123 of John B. Conway's *Functions of One Complex Variable* (Springer-Verlag, 1973). Although the theorem is technically true since when P is false the implication $P \to Q$ is true for any Q, the wording of it seems singularly unfortunate, unless it is intended as a trap for the unwary student.

139. [1976: 68] *Proposed by Dan Pedoe, University of Minnesota.*

ABCD is a parallelogram, and a circle γ touches AB and BC and intersects AC in the points E and F. Then there exists a circle δ which passes through E and F and touches AD and DC.

Prove this theorem without using Rennie's Lemma (see [1976: 65]).

Solution by Kenneth S. Williams, Carleton University.

Let γ touch AB at G and BC at H. Two cases will be considered.

Case 1. GH is parallel to AC (see Figure 1).

Let GH meet AD and CD in K and L, respectively, and let $\angle$BGH $= \alpha$. It is easy to check that each of the ten angles indicated in the figure is equal to α, and so

$$\text{AK} = \text{AG} = \text{CH} = \text{CL}.$$

Now $\overline{\text{AG}}^2 = \text{AE} \cdot \text{AF}$ and $\overline{\text{CH}}^2 = \text{CF} \cdot \text{CE}$, so that $\text{AE} \cdot \text{AF} = \text{CF} \cdot \text{CE}$, that is,

$$\text{AE} \cdot (\text{AE} + \text{EF}) = \text{CF} \cdot (\text{CF} + \text{EF}).$$

Thus

$$(\text{AE} - \text{CF}) \cdot (\text{AE} + \text{CF} + \text{EF}) = 0,$$

and AE = CF. Now ΔKAE is congruent to ΔLCF, so that

$$\angle \text{KEA} = \angle \text{LFC} = \angle \text{FLK},$$

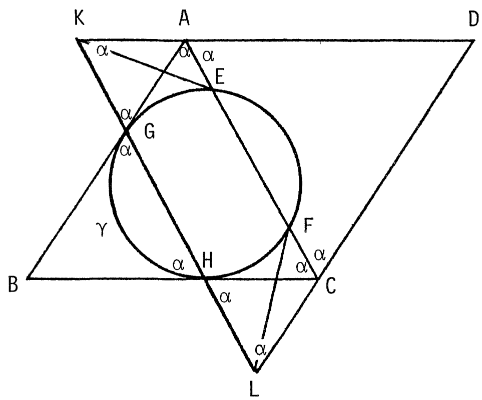


Figure 1

and we conclude that the points K, E, F, L are concyclic, all lying on circle δ, say. Since

$$\overline{\text{AK}}^2 = \overline{\text{AG}}^2 = \text{AE} \cdot \text{AF},$$

it follows that AD touches δ at K. Similarly DC touches δ at L, and δ is the required circle.

Case 2. GH is not parallel to AC (see Figure 2).

Let GH meet AC, AD, CD in J, K, L, respectively. It is clear that Δs JLC and JGA are similar, and so are Δs JHC and JKA; hence

$$\frac{JL}{JG} = \frac{JC}{JA} = \frac{JH}{JK},$$

and so

$$JL \cdot JK = JH \cdot JG.$$

But since J lies outside γ we have

$$JH \cdot JG = JF \cdot JE.$$

Thus $JL \cdot JK = JF \cdot JE$ and so the points L, K, F, E are concyclic, all lying on circle δ, say. If we let $\angle BGH = \beta$, it is easy to check that each of the six angles indicated in the figure is equal to β, and so $AK = AG$, $CH = CL$. Since AG is tangent to γ, we have

$$\overline{AK}^2 = \overline{AG}^2 = AE \cdot AF,$$

and it follows that AD touches δ at K. Similarly DC touches δ at L, and δ is the required circle.

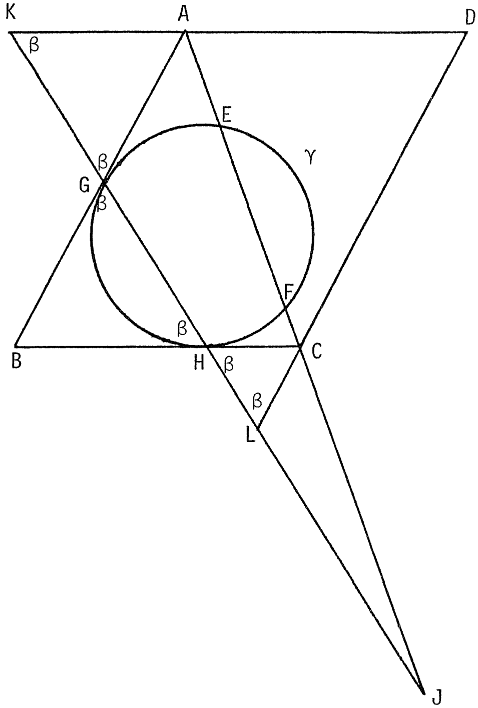


Figure 2

Also solved (trigonometrically) by G.D. KAYE, *Department of National Defence.* *One incorrect solution was received.*

Editor's comment.

As pointed out in [2], this theorem was first announced by the proposer in [1].

REFERENCES

1. Dan Pedoe, The most elementary theorem of Euclidean geometry, *Mathematics Magazine*, Vol. 49 (1976), pp. 40-42.

2. Léo Sauvé, On Circumscribable Quadrilaterals, *Eureka*, Vol. 2 (1976), pp. 63-67.

154. [1976: 110] *Correction.*

The sentence in lines 3 and 4 should read: Corresponding to each r $(1 \le r \le p_n - 1)$ in this list, say $r = p_1^{a_1} \dots p_{n-1}^{a_{n-1}}$, put $p_2^{a_1} \dots p_n^{a_{n-1}}$ in a second row.

* * *

MATHEMATICS AS AN APPETIZER

He sighed, pushing the work away, and took a sheet of paper. He'd always liked to doodle around with numbers, and one of the teachers had taught him a little about algebra. Some of the fellows had called him teacher's pet for that, till he licked them, but it was real interesting, not just like learning multiplication tables. Here you made the numbers and letters do something. The teacher said that if he really wanted to build spaceships when he grew up, he'd have to learn lots of math.

He started drawing some graphs. The different kinds of equations made different pictures. It was fun to see how $x = ky + c$ made a straight line while $x^2 + y_2 = c$ was always a circle. Only how if you changed one of the x's, made it equal 3 instead of 2? What would happen to the y in the meantime? He'd never thought of that before!

He grasped the pencil tightly, his tongue sticking out of the corner of his mouth. You had to kind of sneak up on the x and the y, change one of them just a weeny little bit, and then——

He was well on the way to inventing differential calculus when his mother called him down to breakfast.

POUL ANDERSON, in "Brain Wave", from *A Treasury of Great Science Fiction*, edited by Anthony Boucher, Doubleday, 1959, Vol. 2, p. 9.

He should now learn analytic geometry to work up an appetite for lunch.

* * *

POESY AND NUMBERS

She took me to her elfin grot
And there she wept and sigh'd full sore,
And there I shut her wild wild eyes
With kisses four.

From *La Belle Dame Sans Merci*, by JOHN KEATS

Why four kisses—you will say—why four because I wish to restrain the headlong impetuosity of my Muse—she would have fain said "score" without hurting the rhyme—but we must temper the Imagination as the Critics say with Judgment. I was obliged to choose an even number that both eyes might have fair play: and to speak truly I think two a piece quite sufficient—Suppose I had said seven; there would have been three and a half a piece—a very awkward affair.

From a letter Keats wrote to his brother George in America, quoted in *John Keats*, by Walter Jackson Bate, Harvard University Press, 1964, p. 480.

* * *